CHAPTER

3

Geometric Properties

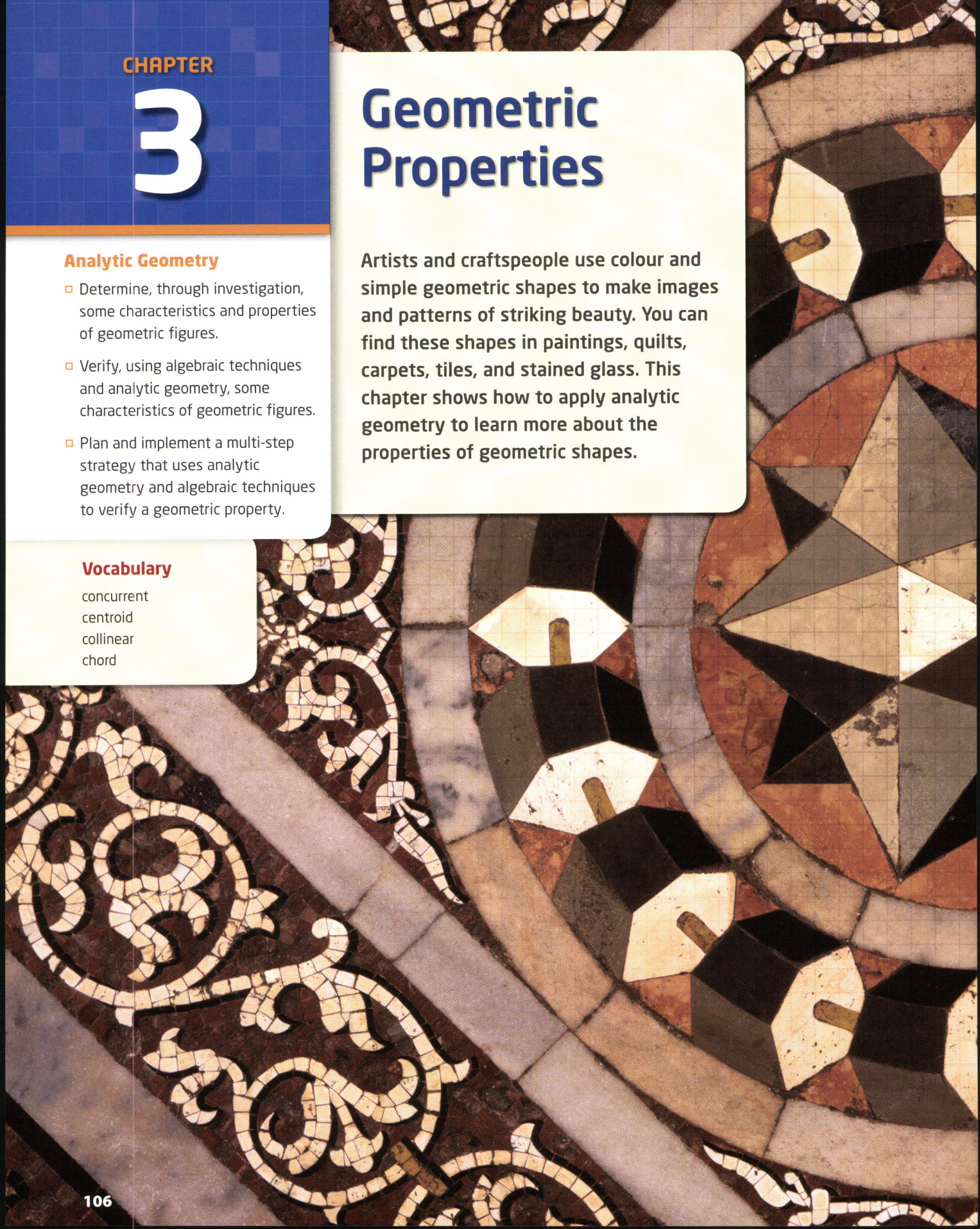
Artists and craftspeople use colour and simple geometric shapes to make images and patterns of striking beauty. You can find these shapes in paintings, quilts, carpets, tiles, and stained glass. This chapter shows how to apply analytic geometry to learn more about the properties of geometric shapes.

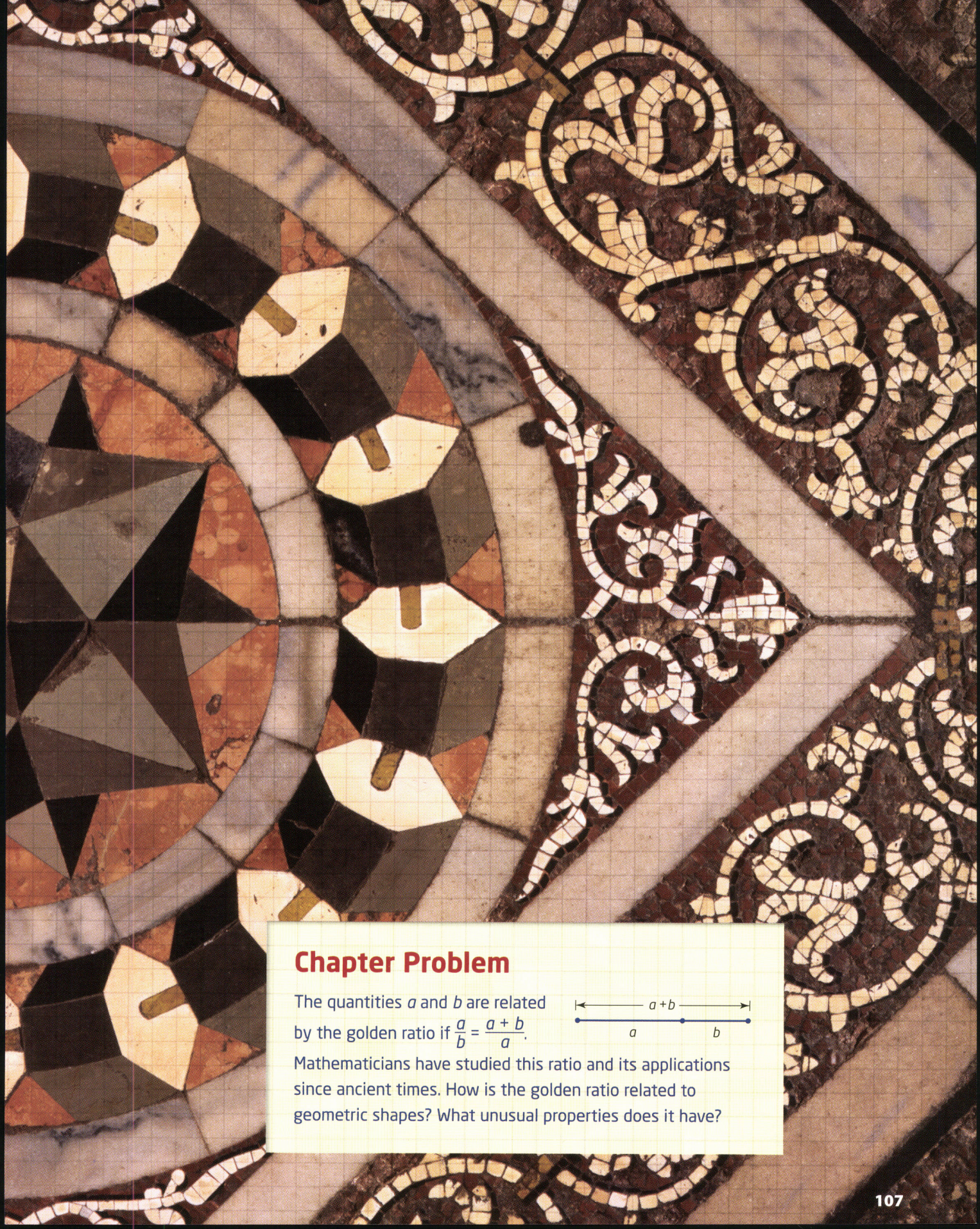
Analytic Geometry

- Determine, through investigation, some characteristics and properties of geometric figures.
- Verify, using algebraic techniques and analytic geometry, some characteristics of geometric figures.
- Plan and implement a multi-step strategy that uses analytic geometry and algebraic techniques to verify a geometric property.

Vocabulary

concurrent
centroid
collinear
chord





Chapter Problem

The quantities a and b are related by the golden ratio if $\frac{a}{b} = \frac{a+b}{a}$.

Mathematicians have studied this ratio and its applications since ancient times. How is the golden ratio related to geometric shapes? What unusual properties does it have?

Get Ready

Length and Midpoint of a Line Segment

You can determine the midpoint and the length of a line segment from the coordinates of its endpoints.

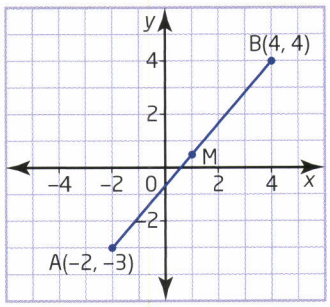
Find the length and midpoint of the line segment joining A(−2, −3) and B(4, 4).

Midpoint of AB:

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{-2 + 4}{2}, \frac{-3 + 4}{2}\right)$$
$$= \left(1, \frac{1}{2}\right)$$

Length of AB:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[4 - (-2)]^2 + [2 - (-3)]^2}$$
$$= \sqrt{6^2 + 5^2}$$
$$= \sqrt{36 + 25}$$
$$= \sqrt{61}$$



1. Find the midpoint of each line segment.

a) CD with C(−1, 5) and D(4, −2)

b) EF with E(−3, 0) and F(5, −2)

c) GH with G(4, 6) and H(4, −3)

d) JK with J(−8, 5) and K(2, 5)

2. Find the length of each line segment.

a) LM with L(−2, 4) and M(3, −3)

b) PQ with P(−4, 0) and Q(3, −2)

c) RS with R(6, 5) and S(6, −4)

d) TU with T(−6, 7) and U(4, 7)

Intersection of Lines

The coordinates of the point of intersection of two lines satisfy the equations of both lines.

Find the point of intersection of the lines defined by $y = -2x + 12$ and $2x - 3y = 12$.

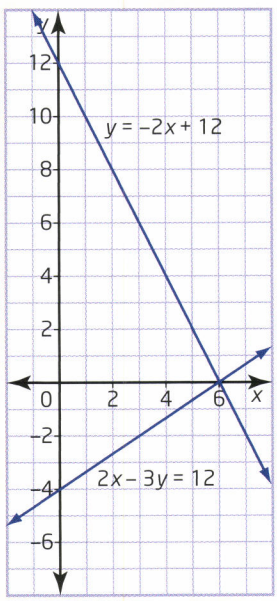
Substitute from the first equation into the second equation.

$$2x - 3y = 12$$
$$2x - 3(-2x + 12) = 12$$
$$2x + 6x - 36 = 12$$
$$8x = 12 + 36$$
$$8x = 48$$
$$x = 6$$

Now, substitute $x = 6$ into the first equation.

$$y = -2x + 12$$
$$= -2(6) + 12$$
$$= 0$$

The point of intersection of the two lines is (6, 0).



3. Find the point of intersection of the lines represented by each pair of equations.

a) $y = x - 4$ and $y = -x + 2$

b) $y = x + 4$ and $y = 2 - x$

c) $y = 2x - 5$ and $y = x - 3$

4. Find the point of intersection of the lines represented by each pair of equations.

a) $2x - 3y = 4$ and $3x - 4y = 5$

b) $2x + y = 5$ and $x - 2y = 10$

c) $2x + y = -5$ and $3x - y = -5$

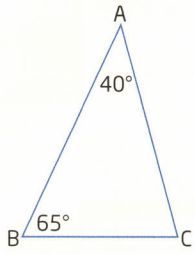
Sum of the Angles in a Triangle

The sum of the angles in a triangle is 180°.

Find the measure of ∠C.

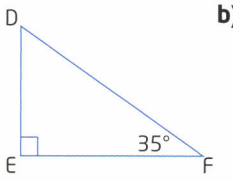
$$\begin{aligned} \angle A + \angle B + \angle C &= 180° \\ \angle C &= 180° - \angle A - \angle B \\ &= 180° - 40° - 65° \\ &= 75° \end{aligned}$$

The measure of ∠C is 75°.

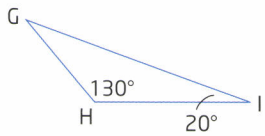


5. Find the measure of the third angle in each triangle.

a)

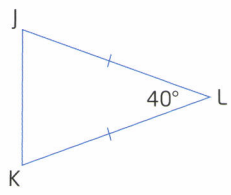


b)

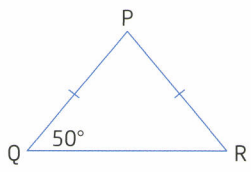


6. Find the measures of the unknown angles in each triangle.

a)



b)

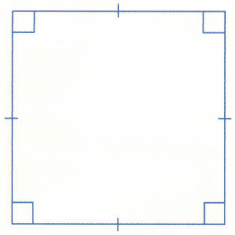


Types of Quadrilaterals

Quadrilaterals are classified by the properties of their sides and angles.

What are the key properties of a square?

A square has four equal sides and four right angles.



7. List the key properties of

a) a rectangle

b) a parallelogram

c) a trapezoid

8. Draw an example of each shape and mark all equal sides and all parallel sides.

a) a rhombus

b) a parallelogram

c) a kite

3.1 Investigate Properties of Triangles

The distribution of an object's mass around its balance point is such that no net force acts to tilt the object. In fact, the object acts like its entire mass is concentrated at a point directly above the balance point. Calculating a balance point accurately can be crucial in the design of aircraft, ships, lift bridges, cranes, and other machinery. Analytic geometry can help find balance points and other properties of geometric shapes.

Investigate

How can you find the balance point of a flat triangular object?

Tools

- ruler
- cardboard
- scissors
- compasses

Method 1: Use Pencil and Paper

1. Draw a large triangle on a sheet of paper. Then, draw an identical triangle on cardboard and cut out this triangle.
2. Move the cardboard triangle around on the tip of a pencil until you find the point where the triangle balances on the pencil. Mark this balance point. Then, mark the corresponding point on the triangle on the sheet of paper.
3. Fold the paper triangle along one of its medians. Where is the balance point relative to this fold?
4. Explain how you know that the median bisects the area of the triangle. Will the cardboard triangle balance on the edge of a ruler if the median is aligned with the edge? Explain your reasoning. Try balancing the cardboard triangle on the edge of a ruler.
5. Fold the paper triangle in half along a different median. What do you notice about the balance point?
6. Fold the paper triangle along the third median. What do you notice about the creases from the three folds? Check whether your classmates get the same result.

7. Reflect

a) Are the medians of your triangle **concurrent**? Do you think that all triangles have this property? Explain your reasoning.

b) How is the balance point of your triangle related to the **centroid** of the triangle? Do you think that this relationship applies for all triangles? Explain.

concurrent

- meeting at a single point

centroid

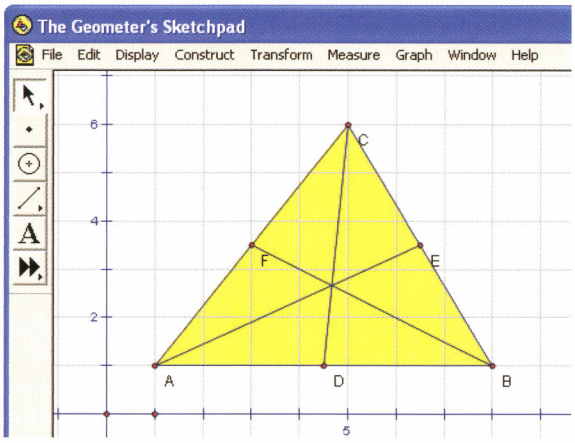
- the point where the three medians of a triangle intersect

Method 2: Use *The Geometer's Sketchpad*®

Tools

- computer with *The Geometer's Sketchpad*®

1. Construct any triangle. Then, construct the three medians of the triangle. Are the medians **concurrent**?



2. Observe the intersection of the medians while you drag each of the vertices of the triangle to change the shape of the triangle.

3. Reflect Does every triangle have a **centroid**, a single point where all three medians intersect? Explain your reasoning.

4. Construct and measure the areas of the two triangles that have median AE as their base. Compare these areas. Then, construct the altitude to AE for both triangles. Compare the lengths of these altitudes.

5. Compare the areas and altitudes of the two triangles that have median BF as their base. Then, compare the areas and altitudes of the two triangles that have median CD as their base.

6. Drag the vertices of the original triangle around the screen. Do the relationships among the areas and altitudes of the smaller triangles change?

Technology Tip

When there are overlapping triangles, you can change which interior is selected by clicking on it again.

7. Reflect

a) What property does the median of a triangle have? Explain how you know that all medians have this property.

b) Will a flat triangular object balance if it is placed with one of its medians along the edge of a ruler? Explain.

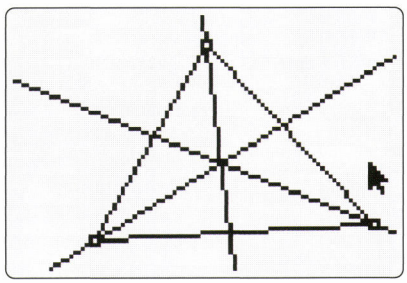
c) Where is the balance point of a flat triangular object? Explain your reasoning.

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

Method 3: Use a Graphing Calculator

1. Start the Cabri® Jr. application. Choose **Triangle** from the **F2** menu, and draw a large triangle. Choose **Midpoint** from the **F3** menu, and construct the midpoint of each side of the triangle. Choose **Segment** from the **F2** menu, and draw the medians of the triangle by joining each vertex to the midpoint of the opposite side. Are the medians **concurrent**?



2. Move the cursor to one of the vertices of the triangle. Press ALPHA. Observe the intersection of the medians while you drag the vertex of the triangle to change the shape of the triangle. Try dragging the other vertices as well.

3. **Reflect** Does every triangle have a **centroid**, a single point where all three medians intersect? Explain your reasoning.

4. Construct a new triangle on top of the original triangle by selecting the endpoints of a median and one of the other vertices of the original triangle. Measure the area of the new triangle. Then, construct the other triangle that has the median as its base, and measure the area of this triangle. Compare the areas of the two smaller triangles. How are the altitudes to the common base of the two triangles related?

Technology Tip

Measure the area of each triangle before constructing the next triangle on top of it. Once a triangle is covered, you cannot select it for measurement.

5. Compare the areas and altitudes of the two triangles that have the second median as their base. Then, compare the areas and altitudes of the two triangles that have the third median as their base.

6. Drag the vertices of the original triangle around the screen. Do the relationships among the areas and altitudes of the smaller triangles change?

7. **Reflect**

   a) What property does the median of a triangle have? Explain how you know that all medians have this property.

   b) Will a flat triangular object balance if it is placed with one of its medians along the edge of a ruler? Explain.

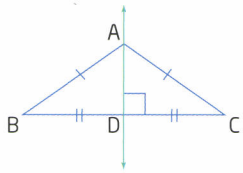
   c) Where is the balance point of a flat triangular object? Explain your reasoning.

Example Median of an Isosceles Triangle

What are the properties of the median from the vertex between the equal sides of an isosceles triangle?

Solution

Draw any isosceles triangle. Label the equal sides AB and AC. Let D be the midpoint of BC.



The right bisector of a line segment includes all points that are equidistant from the endpoints of the line segment. Since AB = AC, vertex A lies on the right bisector of BC. The midpoint, D, also lies on the right bisector of BC. So, AD is perpendicular to BC.

Since the median from vertex A is perpendicular to BC, this median is also the altitude from vertex A.

$\triangle ABC$ and $\triangle ACD$ have side AD in common. The other corresponding sides of these two triangles are equal since AB = AC and D is the midpoint of BC. Therefore, $\triangle ABD$ and $\triangle ACD$ are congruent, and $\angle BAD = \angle CAD$. So, AD bisects $\angle BAC$.

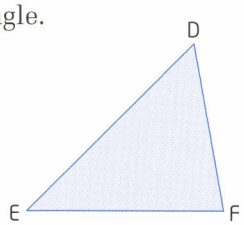
Therefore, the median from the vertex between the equal sides of an isosceles triangle coincides with the altitude to the vertex and bisects the angle at the vertex.

Key Concepts

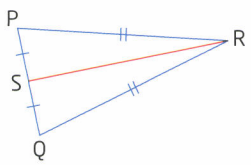
- The medians of a triangle meet at a single point, the centroid.
- Each median bisects the area of the triangle.
- The median from the vertex between the equal sides of an isosceles triangle coincides with the altitude to the vertex and bisects the angle at the vertex.
- You can use both pencil-and-paper techniques and geometry software to investigate the properties of geometric shapes.

Communicate Your Understanding

C1 Describe two methods for drawing a median of this triangle.

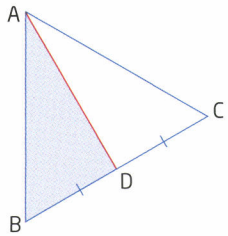


C2 Explain how you know that this diagram has three pairs of equal angles.

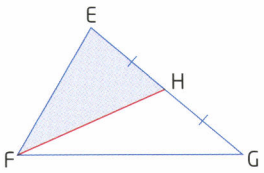


Practise

1. The area of $\triangle ABC$ is 12 square units. Find the area of $\triangle ABD$.



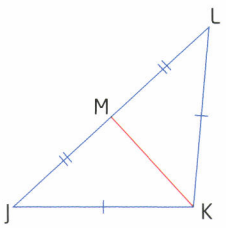
2. The area of $\triangle EFH$ is 30 square units. Find the area of $\triangle EFG$.



For help with questions 3 and 4, see the Example.

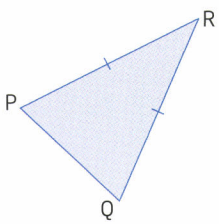
3. a) Which line segments are perpendicular in $\triangle JKL$?

b) Which angles are equal?



4. Which of the following coincide in $\triangle PQR$?

- the bisector of $\angle P$
- the bisector of $\angle R$
- the altitude from vertex Q
- the altitude from vertex R
- the right bisector of side PQ
- the median from vertex P



Connect and Apply

5. a) Draw any isosceles triangle. Then, construct the altitude from the vertex between the equal sides.

b) List three properties of this altitude that an altitude of a scalene triangle does not have.

c) Verify these properties by measuring your drawing.

6. Steve says, "If a median of a triangle is also an angle bisector, the triangle must be equilateral." Is he correct? Justify your answer.

7. a) Draw any right triangle. Find and label the midpoint of the hypotenuse.

b) Measure the distance from this midpoint to each of the vertices of the triangle. How are these distances related?

c) Investigate whether the same relationship applies for all right triangles. Describe your method and your findings.

8. List at least six properties of equilateral triangles. Explain how you know that every equilateral triangle has each of these properties.

9. Alana claims, "In an equilateral triangle, the angle bisectors and the right bisectors of the sides all meet at the same point." Do you agree with Alana's claim? Explain, using a diagram.

10. a) Draw any triangle and all three of its medians. The centroid divides each median into two sections. Compare the lengths of the sections of each median.

b) Investigate whether the centroid in every triangle divides each median in the same ratio. Outline your method and describe your findings.

c) Explain how you can use your findings in part b) to find the balance point of a flat triangular object.

11. Use Technology

a) Construct any $\triangle ABC$. Then, construct the midpoint D of side AB and the midpoint E of side AC.

b) Compare the slope and the length of line segment DE to those of side BC.

c) Investigate whether the relationships between DE and BC depend on the shape of the triangle. Describe your findings.

12. a) Draw a large triangle on a sheet of paper. Then, draw the angle bisector at each vertex. Are the three angle bisectors of your triangle concurrent?

b) Repeat part a) with different types of triangles. Does every triangle have an **incentre,** a point where all three angle bisectors meet? Explain your reasoning.

c) Put the point of a set of compasses at the incentre of a triangle, and adjust the compasses so that the pencil just touches one side of the triangle. Draw a circle with this radius. What property does the incentre of this triangle have?

d) What property does the incentre of every triangle have? Justify your answer.

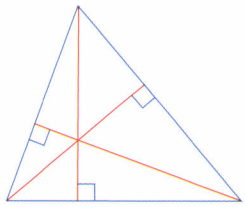
13. Use Technology Use geometry software to investigate the properties of the angle bisectors of a triangle. Outline your method and describe your findings.

14. a) Investigate whether every triangle has a **circumcentre**, a point where all the right bisectors of the sides meet. Describe your method and your results.

b) How are the distances from the circumcentre to the vertices related? Do all triangles have this property? Justify your answer.

c) A company with plants in Hamilton, Oshawa, and Barrie wants to set up a distribution centre that is roughly equidistant from the three plants. How can the company use the properties of triangles to find a suitable location?

15. In this triangle, the altitudes from the three vertices meet at a single point, called the **orthocentre**. Investigate whether the altitudes intersect at a single point in all triangles. Outline your method and describe your findings.



16. Use a triangle to estimate the geographical centre of Canada.

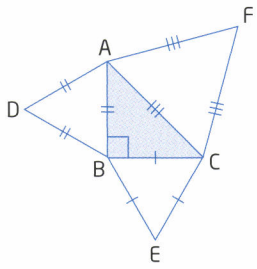
a) Trace the outline of Canada from a map onto a sheet of paper. Draw a triangle on your outline to approximate the shape of Canada as closely as you can.

b) Find the centroid of the triangle. Then, draw the angle bisectors of the triangle and find their point of intersection. Also find the point of intersection of the right bisectors of the sides and the point of intersection of the altitudes of the triangle. Which of these four points gives the best approximation for the geographic centre of Canada? Justify your choice.

c) Find the town or city closest to the centre you chose in part b). Compare your estimate with those made by your classmates.

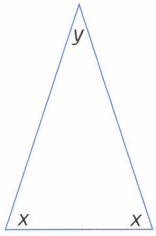
17. a) Investigate how the areas of equilateral triangles constructed on the sides of a right triangle are related. Outline your method and describe your findings.

b) Describe another method that you could use to compare the areas of the triangles.



18. Chapter Problem

a) Let x represent the measure of the base angles of an isosceles triangle and let y represent the angle at the vertex between the two equal sides. Find the values of x and y when $x = 2y$.



b) Draw a large triangle that has the angle measures that you calculated in part a). Measure the lengths of the sides of your triangle.

c) Calculate the ratio $\frac{a}{b}$, where a is the length of one of the equal sides and b is the length of the base of the isosceles triangle.

d) The golden ratio, denoted by the Greek letter φ (phi), is equal to $\frac{1 + \sqrt{5}}{2}$.

Compare the decimal value for φ to the ratio you calculated in part c).

e) A golden triangle is an isosceles triangle with side lengths that correspond to the golden ratio. Is the triangle you drew in part b) a golden triangle?

f) Draw an isosceles triangle with the angle at the vertex between the two equal sides three times as large as the base angles. Is this triangle a golden triangle? Explain.

g) Draw an isosceles triangle with the angle at the vertex between the two equal sides four times as large as the base angles. Is this triangle a golden triangle?

Extend

19. Does the point of intersection of the angle bisectors of a triangle (the incentre) always lie inside the triangle? Justify your answer.

20. When will the point of intersection of the right bisectors of the sides of a triangle (the circumcentre) lie

a) outside the triangle?

b) on a side of the triangle?

Justify your answers using diagrams.

21. Investigate how the locations of the centroid, the orthocentre, and the circumcentre of a triangle are related. Describe your method and your findings.

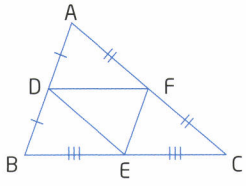
22. Investigate whether the point of intersection of the altitudes of a triangle (the orthocentre) can lie

a) outside the triangle

b) on the triangle

Outline your method and describe your findings.

23. Math Contest Show that the medians of $\triangle ABC$ and the medians of $\triangle DEF$ all meet at a single point.



3.2 Verify Properties of Triangles

Since triangular frames are strong and simple to make, they are widely used to strengthen buildings and other structures. This section applies analytic geometry to verify the properties of specific triangles.

Investigate

Tools

- grid paper

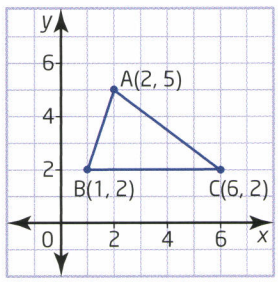
How can you verify the properties of an isosceles triangle?

Method 1: Use Pencil and Paper

1. Draw the triangle with vertices A(2, 5), B(1, 2), and C(6, 2). Use the coordinates of the vertices to verify that $\triangle ABC$ is isosceles. Which angles of $\triangle ABC$ are equal?

2. Describe how you could fold your drawing of $\triangle ABC$ to confirm that it is isosceles.

3. Find the coordinates of the midpoint, D, of side AB. Draw the median from vertex C. Use slopes to verify that this median is perpendicular to AB.



4. Explain how you can use congruent triangles to verify that CD is perpendicular to AB. Describe two different ways to show that $\triangle ADC$ and $\triangle BDC$ are congruent.

5. Use angle sums to verify that CD bisects $\angle ACB$.

6. Describe another way to show that CD bisects $\angle ACB$.

7. **Reflect** How are the median, the altitude, and the angle bisector at vertex C related? How are these line segments related to the right bisector of AB? Do you think all isosceles triangles have these properties? Explain your reasoning.

Tools

- computer with *The Geometer's Sketchpad®*

Technology Tip

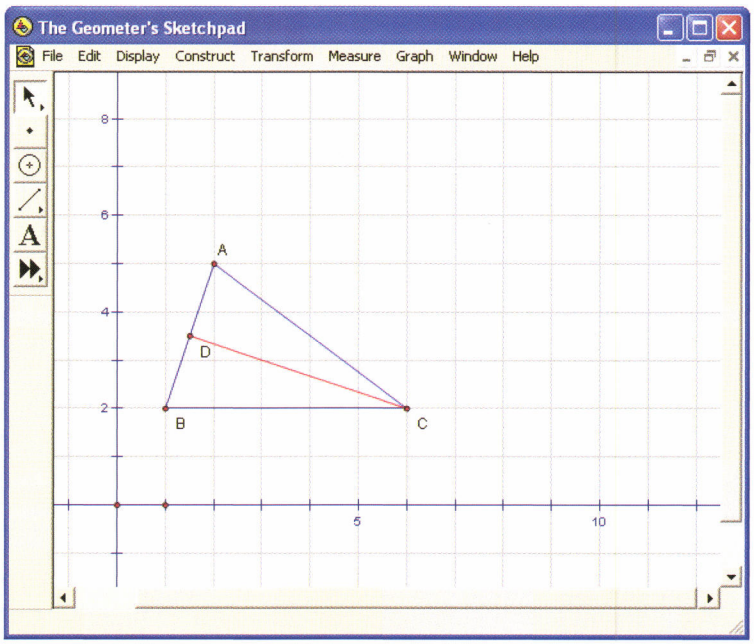
You can also place the vertices by choosing **Plot Points** from the **Graph** menu and typing the coordinates.

Method 2: Use *The Geometer's Sketchpad®*

1. Choose **Show Grid** from the **Graph** menu. Then, choose **Snap Points**. Move the origin so that most of the work area is in the first quadrant. Construct the triangle with vertices A(2, 5), B(1, 2), and C(6, 2).

2. Measure the length of each side to verify that $\triangle ABC$ is isosceles. Which angles of $\triangle ABC$ are equal?

3. Construct the midpoint, D, of side AB. Then, construct the median from vertex C. Measure $\angle ADC$. What can you conclude about the median CD?



4. Measure and compare $\angle ACD$ and $\angle BCD$.

5. Turn off **Snap Points**. Drag a vertex of $\triangle ABC$ to a new location. Move one of the other vertices until AC = BC. Observe the measures of $\angle ADC$, $\angle ACD$, and $\angle BCD$ as you repeat this process for various locations of the vertices.

6. **Reflect** How are the median, the altitude, and the angle bisector at vertex C related? How are these line segments related to the right bisector of AB? Do you think all isosceles triangles have these properties? Explain your reasoning.

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

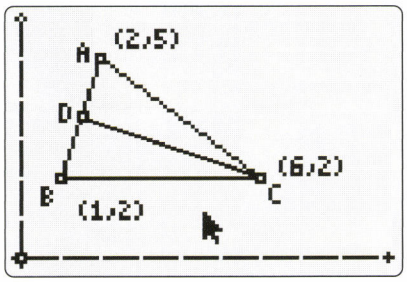
Method 3: Use a Graphing Calculator

1. Start the Cabri® Jr. application. If the axes are not displayed, choose **Hide/Show** from the **F5** menu; then, choose **Axes** from the submenu. Move the origin so that most of the work area is in the first quadrant.

2. Draw the triangle with vertices A(2, 5), B(1, 2), and C(6, 2). Choose **Coord. & Eq.** from the **F5** menu. Then, select the vertices to display their coordinates. To adjust a vertex, move the cursor to it, press ALPHA, and use the arrow keys to reposition the vertex.

3. Choose **Measure** from the **F5** menu, and then choose **D. & Length**. Measure the length of each side to verify that the triangle is isosceles. Which angles of $\triangle ABC$ are equal?

4. Choose **Midpoint** from the **F3** menu and construct the midpoint, D, of side AB. Then, choose **Segment** from the **F2** menu and construct the median from vertex C. Measure $\angle ADC$. What can you conclude about the median DC?



5. Measure and compare $\angle ACD$ and $\angle BCD$.

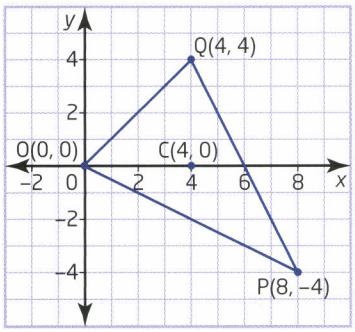
6. Drag a vertex of $\triangle ABC$ to a new location. Move one of the other vertices until AC = BC. Observe the measures of $\angle ADC$, $\angle ACD$, and $\angle BCD$ as you repeat this process for various locations of the vertices.

7. **Reflect** How are the median, the altitude, and the angle bisector at vertex C related? How are these line segments related to the right bisector of AB? Do you think all isosceles triangles have these properties? Explain your reasoning.

Example 1 Centroid of a Triangle

a) Verify that C(4, 0) is the centroid of $\triangle OPQ$.

b) Verify that the centroid divides each median in a 2:1 ratio.

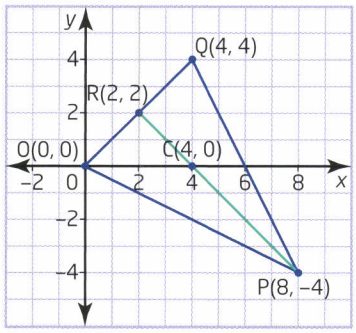


Solution

a) The centroid of a triangle is the point of intersection of the three medians. Verify that C(4, 0) is the centroid by showing that the coordinates of this point satisfy the equations for the lines that include the three medians.

Start by using the midpoint formula to find the coordinates of the midpoint, R, of side OQ.

$$R(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{0 + 4}{2}, \frac{0 + 4}{2}\right)$$
$$= (2, 2)$$



Use these coordinates to find the slope of the median from vertex P(8, −4).

$$m_{PR} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - (-4)}{2 - 8}$$
$$= \frac{6}{-6}$$
$$= -1$$

I could also substitute the coordinates of points P and R into $y = mx + b$ to get two equations with m and b as the unknowns. Solving this system of equations gives values for m and b.

Use this slope and the coordinates of point R to find the y-intercept.

$y = mx + b$
$2 = -1(2) + b$
$2 = -2 + b$
$4 = b$

An equation for the line that includes the median PR is $y = -x + 4$.

Now, substitute the coordinates of point C(4, 0) into each side of the equation.

L.S. $= y$
$= 0$

R.S. $= -x + 4$
$= -4 + 4$
$= 0$

Since the coordinates of point C satisfy the equation and the point lies within the triangle, C(4, 0) lies on the median PR.

Next, find the midpoints, S and T, of sides OP and PQ.

$$S(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{0 + 8}{2}, \frac{0 + (-4)}{2}\right)$$
$$= (4, -2)$$

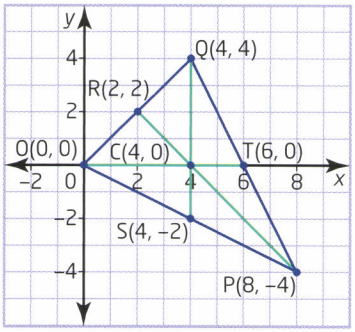
$$T(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{8 + 4}{2}, \frac{-4 + 4}{2}\right)$$
$$= (6, 0)$$

The method used for the median PR will also show that point C lies on the medians OT and QS. However, examining the coordinates of the points and the diagram of △OPQ reveals a shortcut for the medians OT and QS.

Since S(4, −2), C(4, 0), and Q(4, 4) all have the same x-coordinate, these three points lie on the vertical line with equation $x = 4$. Therefore, point C lies on the median QS. Similarly, O(0, 0), C(4, 0), and T(6, 0) are **collinear** since these points all lie on the x-axis. So, point C also lies on the median OT.

collinear

- lying on the same line



Since the point C(4, 0) lies on all three medians, it is the centroid of △OPQ.

b) To find the ratio of the parts of a median on either side of the centroid, use the length formula to find the length of each part.

For the median PR, compare the lengths of PC and RC.

$$\begin{aligned} PC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(8 - 4)^2 + (-4 - 0)^2} \\ &= \sqrt{4^2 + (-4)^2} \\ &= \sqrt{4^2 \times 2} \\ &= \sqrt{4^2} \times \sqrt{2} \\ &= 4\sqrt{2} \end{aligned}$$

$$\begin{aligned} RC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - 4)^2 + (2 - 0)^2} \\ &= \sqrt{2^2 + 2^2} \\ &= \sqrt{2^2 \times 2} \\ &= \sqrt{2^2} \times \sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

To move a factor out from under a square root sign, I take the square root of the factor:

$\sqrt{nx} = \sqrt{n} \times \sqrt{x}$

PC is twice the length of RC.

Since the median OT is a horizontal line segment, find the lengths of OC and TC by simply comparing the x-coordinates of the endpoints. Subtract the lesser x-coordinate from the greater one.

$$\begin{aligned} OC &= x_2 - x_1 \\ &= 4 - 0 \\ &= 4 \end{aligned} \qquad \begin{aligned} TC &= x_1 - x_2 \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

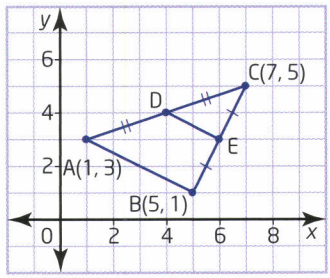
Since QS is a vertical line segment, compare y-coordinates to find the lengths of QC and SC.

$$\begin{aligned} QC &= y_1 - y_2 \\ &= 4 - 0 \\ &= 4 \end{aligned} \qquad \begin{aligned} SC &= y_2 - y_1 \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

The centroid C(4, 0) divides each of the medians of △OPQ into a 2:1 ratio.

Example 2 Midpoints of the Sides of a Triangle

In $\triangle ABC$, D is the midpoint of side AC and E is the midpoint of side BC.



a) Verify that line segment DE is parallel to side AB.

b) Verify that line segment DE is half the length of side AB.

c) Use geometry software to check your calculations in parts a) and b).

Solution

a) First, use the coordinates of the vertices to find the coordinates of the midpoints D and E.

$$\begin{aligned} \mathrm{D}(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{1 + 7}{2}, \frac{3 + 5}{2}\right) \\ &= (4, 4) \end{aligned} \qquad \begin{aligned} \mathrm{E}(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{5 + 7}{2}, \frac{1 + 5}{2}\right) \\ &= (6, 3) \end{aligned}$$

Now, use the coordinates of points A, B, D, and E to compare the slope of AB to the slope of DE.

$$\begin{aligned} m_{\mathrm{AB}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - 3}{5 - 1} \\ &= \frac{-2}{4} \\ &= -\frac{1}{2} \end{aligned} \qquad \begin{aligned} m_{\mathrm{DE}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3 - 4}{6 - 4} \\ &= -\frac{1}{2} \end{aligned}$$

Since the slopes are the same, AB is parallel to DE.

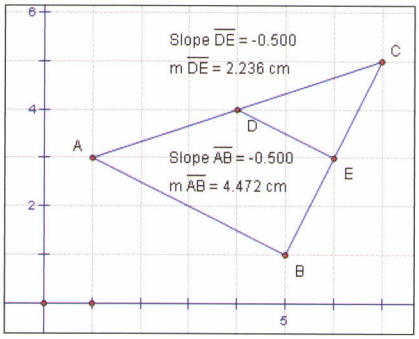
b) Use the distance formula to compare the length of DE to the length of AB.

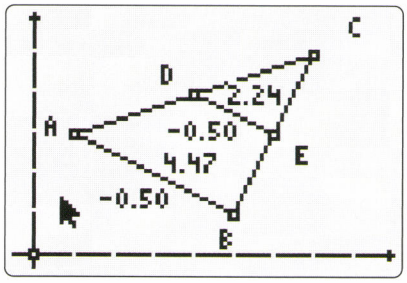
$$\begin{aligned} \mathrm{DE} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - 4)^2 + (3 - 4)^2} \\ &= \sqrt{2^2 + (-1)^2} \\ &= \sqrt{5} \end{aligned} \qquad \begin{aligned} \mathrm{AB} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5 - 1)^2 + (1 - 3)^2} \\ &= \sqrt{4^2 + (-2)^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \\ &= \sqrt{4 \times 5} \\ &= \sqrt{4} \times \sqrt{5} \\ &= 2\sqrt{5} \end{aligned}$$

Therefore, DE is half the length of AB.

c) Use either *The Geometer's Sketchpad*® or Cabri® Jr. to construct the triangle with vertices A(1, 3), B(5, 1) and C(7, 5). Construct the midpoint of AC and the midpoint of BC. Label these midpoints D and E, respectively. Then, construct the line segment DE. Use the software's measurement tools to find the slope and length of DE and of AB.

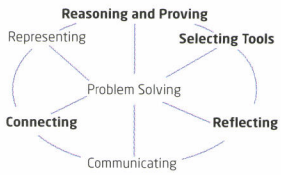
Comparing these measurements shows that the slopes are equal and AB = 2DE. Therefore, DE is parallel to AB and half the length of AB.





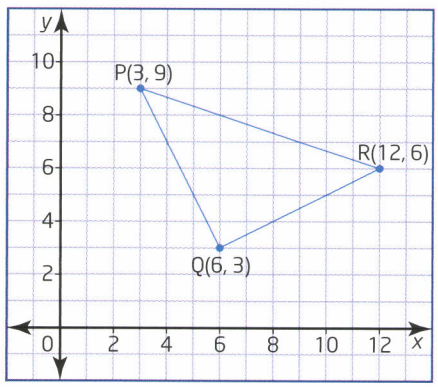
Key Concepts

- You can use the midpoint, length, and slope formulas to verify properties of specific triangles.
- Sometimes, there are several different ways to verify a property of a given triangle.
- Often, you can use a shortcut for calculations involving horizontal or vertical line segments.
- The centroid of a triangle divides each median into two parts, with one part twice the length of the other.
- The line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

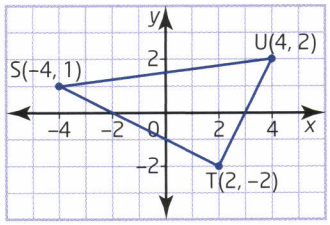


Communicate Your Understanding

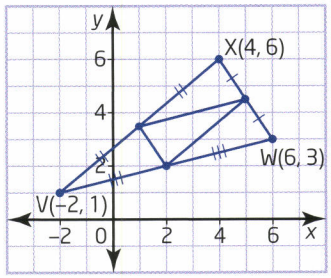
C1 Describe two different ways to verify that △PQR is isosceles.



C2 Outline how to use analytic geometry to verify that the part of the median on one side of the centroid of $\triangle STU$ is twice the length of the part on the other side.



C3 Describe how to use geometry software to verify that each line segment within $\triangle VWX$ is parallel to one of the sides of the triangle.

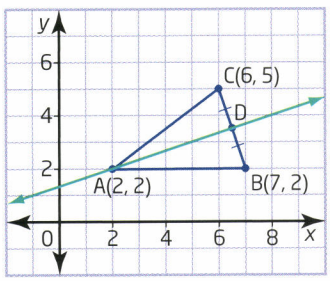


Practise

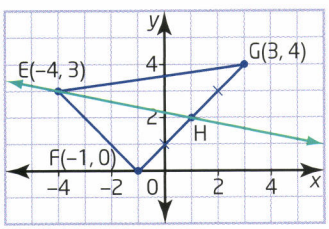
For help with question 1, see Example 1.

1. Determine an equation for the line shown with each triangle.

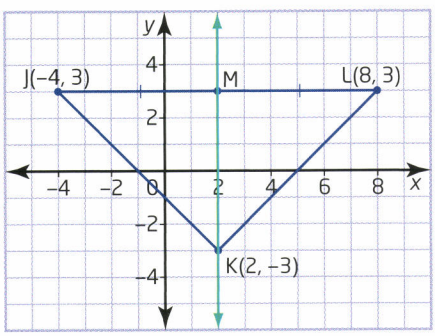
a)



b)

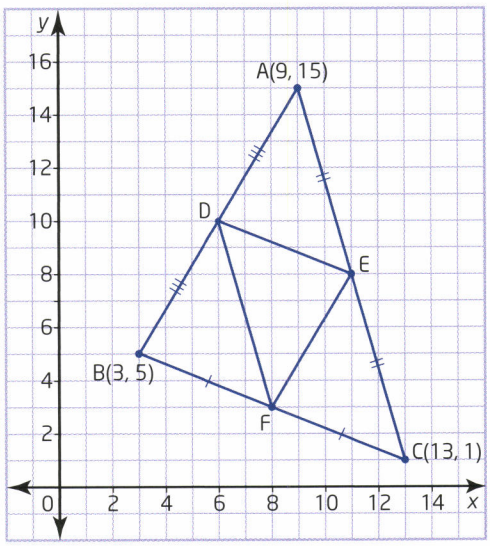


c)



For help with questions 2 and 3, see Example 2.

2. a) Verify that DE and BC are parallel.

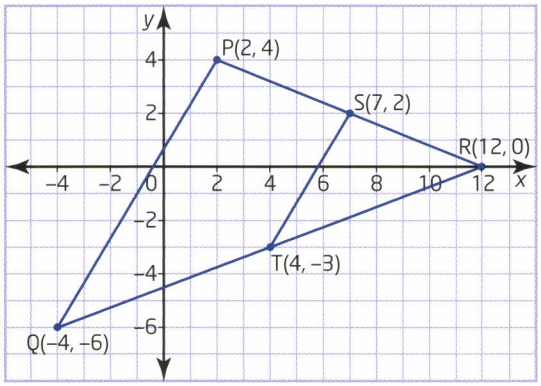


b) List the other line segments that are parallel.

c) Verify that DE = BF.

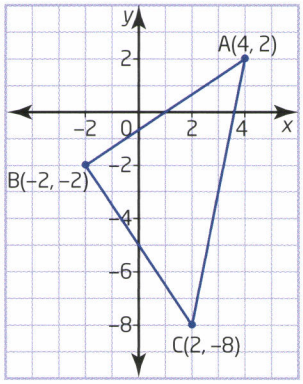
d) List the other line segments that have equal lengths.

3. Verify that PQ is twice the length of ST.



Connect and Apply

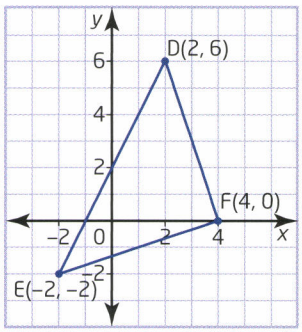
4. a) Verify that △ABC is isosceles.



b) Verify that the median from vertex B is also an altitude of the triangle.

5. Use Technology Use geometry software to verify your answers to question 4.

6. a) Find the lengths of the sides of △DEF.



b) Find the slopes of the sides of the triangle.

c) Classify △DEF. Explain your reasoning.

7. a) Draw the triangle with vertices J(8, 8), K(−5, −5), and L(5, −7). What type of triangle does △JKL appear to be?

b) Use analytic geometry to verify your classification of △JKL.

c) Determine the perimeter of the triangle.

d) Determine the area of the triangle.

8. Use Technology Use geometry software to verify your answers to question 7.

9. a) Draw the triangle with vertices P(−12, 6), Q(4, 0), and R(−8, −6).

b) Determine the coordinates of S, the midpoint of PR, and T, the midpoint of PQ.

c) Verify that ST is parallel to QR.

d) Verify that ST is half the length of QR.

10. a) Draw the triangle with vertices A(3, 4), B(−2, 0), and C(5, 0). Find the midpoint of each side, and label these midpoints D, E, and F.

b) Verify that △DEF is similar to △ABC. Find the ratio of the lengths of corresponding sides of these triangles.

c) Verify that the area of △ABC is four times the area of △DEF.

d) How is the ratio of the lengths of corresponding sides related to the ratio of the areas of △ABC and △DEF?

11. Use Technology Use geometry software to verify your answers to question 10.

12. A landscape architect is drawing plans for a rigid triangular canopy to provide shade in a courtyard. On the drawing, the vertices of the canopy are O(0, 0), P(10, 0), and Q(2, 12). A single pole will support the canopy.

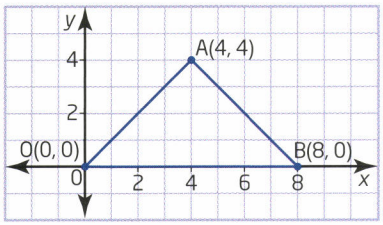
a) Verify that the triangular canopy has a centroid.

b) Explain why the centroid is a good location for attaching the canopy to the support pole.

13. a) Verify that the triangle with vertices J(1, 2), K(−3, −1), and L(0, −5) is an isosceles right triangle.

b) Describe another method that you could use to answer part a).

14. a) Determine the equations of the right bisectors of the sides of △OAB.



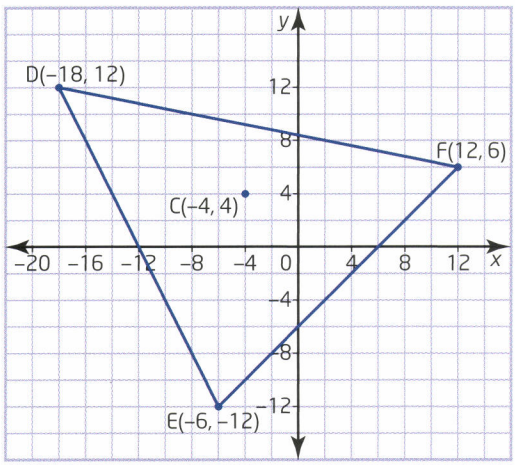
b) Determine the coordinates of the circumcentre, the point of intersection of the right bisectors of the sides.

c) What kind of triangle is △OAB? Justify your answer.

d) Describe the location of the circumcentre of this triangle.

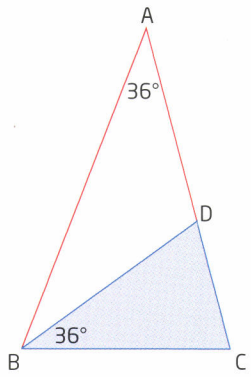
15. a) Show that the right bisectors of the sides of △DEF all intersect at point C(−4, 4), the circumcentre of the triangle.

b) Verify that point C is equidistant from the three vertices of △DEF.



16. Chapter Problem

a) Draw a large △ABC with ∠A = 36° and ∠B = ∠C = 72°. Bisect ∠B. Label the intersection of the angle bisector with side AC as point D.



b) Verify that both △ABC and △BCD are golden triangles. Explain how you know that these two triangles are similar.

c) Construct the bisector of ∠C, and label the intersection with BD as point E. Then, bisect ∠CDE. Continue this process to produce a series of smaller and smaller golden triangles.

d) Draw a smooth curve through points A, B, C, D, E, and so on. This curve is a golden spiral.

e) Is this golden spiral a fractal? Explain.

Achievement Check

17. Verify that the points A(2, 1), B(8, 5), and C(−1, −1) are collinear using

a) slopes

b) an equation of a line

c) lengths

Extend

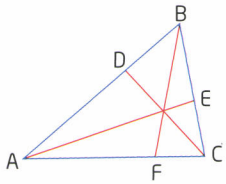
18. A triangle has vertices P(a, b), Q(c, d), and R(e, f).

a) Determine the coordinates of S and T, the midpoints of PQ and PR, respectively.

b) Verify that ST is parallel to QR

c) Verify that ST is half the length of QR.

19. A **cevian** is any line segment that joins a vertex of a triangle to a point on the opposite side. In 1678, the Italian mathematician Giovanni Ceva published a theorem about concurrent cevians in a triangle. Ceva's theorem states that $\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = 1$, where AE, BF, and CD are concurrent cevians in $\triangle ABC$.

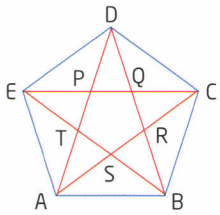


a) Draw any three concurrent cevians in a large scalene triangle. Measure the line segments to verify that Ceva's theorem applies for your triangle.

b) Verify that Ceva's theorem applies for the medians of any triangle.

c) **Use Technology** Use geometry software to verify that Ceva's theorem applies for all triangles. Outline your method.

20. ABCDE is a regular pentagon. Which of the triangles formed by the diagonals of pentagon ABCDE are golden triangles?



21. **Math Contest** The golden ratio is $\varphi = \frac{1 + \sqrt{5}}{2}$. Show that $\varphi^2 = \varphi + 1$.

22. **Math Contest** How many different three-letter arrangements can you make using the letters in the word *MOLLY*?

A 6

B 30

C 33

D 60

E 120

Making Connections

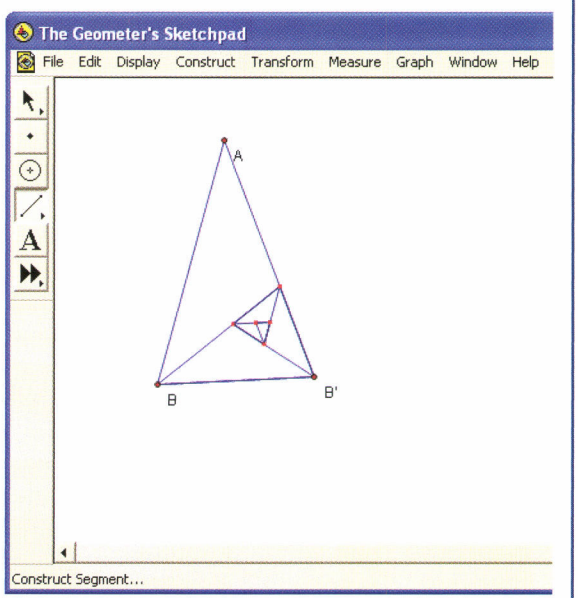
You can use *The Geometer's Sketchpad*® to generate nested golden triangles.

Construct a long line segment AB. Select point A, and choose **Mark Center** from the **Transform** menu. Select point B, and choose **Rotate** from the **Transform** menu. Change the rotation angle to 36°. Construct line segments from A to B′ and B to B′.

Select points A and B, in that order. Choose **Iterate** from the **Transform** menu. Map A onto B and B onto B′. Press + or − to set the number of iterations to at least 5. Click on **Iterate**.

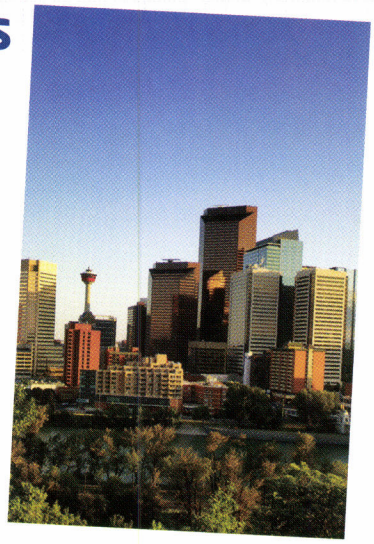
Drawing a smooth curve through the vertices of the nested golden triangles produces a golden spiral. You can draw this spiral by hand on a printout. There are Web sites with software for drawing golden spirals.

WWW Go to www.mcgrawhill.ca/links/principles10 and follow the links to find sites that generate golden triangles and golden spirals.



3.3 Investigate Properties of Quadrilaterals

Rectangular shapes dominate the skylines of most cities in North America. This section investigates the properties of rectangles and other types of quadrilaterals.



Investigate

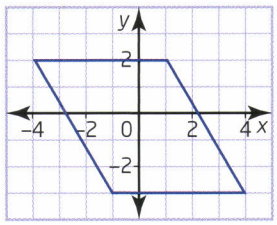
What properties does a parallelogram have?

Tools

- grid paper
- ruler
- compasses
- protractor

Method 1: Use Pencil and Paper

1. Draw a large parallelogram on grid paper. Describe your method and explain how you know that it produces a parallelogram.



2. Draw the diagonals of the parallelogram. Describe the location of the point where the diagonals intersect.

3. Make a conjecture about how the point of intersection of the diagonals divides each diagonal.

4. Fold your parallelogram drawing to test your conjecture. Describe your findings. Compare your results with those of several classmates.

5. **Reflect** What can you conclude about the diagonals of the parallelograms drawn by you and your classmates? Do you think that this conclusion applies to all parallelograms? Explain your reasoning.

6. Mark the midpoint of each side of the parallelogram. Join the midpoints of adjacent sides to form a quadrilateral within the parallelogram. What properties does the new quadrilateral appear to have?

7. Make a conjecture about the type of quadrilateral formed by joining the midpoints of adjacent sides of a parallelogram.

8. Use measurements or paper-folding to determine whether your conjecture is true. Describe your findings, and compare them with those of several classmates.

9. **Reflect** What can you conclude about the midpoints of the sides of the parallelograms drawn by you and your classmates? Do you think that this conclusion applies to all parallelograms? Explain.

Method 2: Use *The Geometer's Sketchpad*®

Tools

- computer with *The Geometer's Sketchpad*®

1. Turn on the grid display and automatic labelling of points.

2. Construct line segment AB and point C above it. Connect B to C with a line segment. Select point C and line segment AB. Choose **Parallel Line** from the **Construct** menu. Select point A and line segment BC. Then, choose **Parallel Line** from the **Construct** menu again.

3. Select the two lines that you constructed, and choose **Intersection** from the **Construct** menu. Select the two lines again, and choose **Hide Parallel Lines** from the **Display** menu.

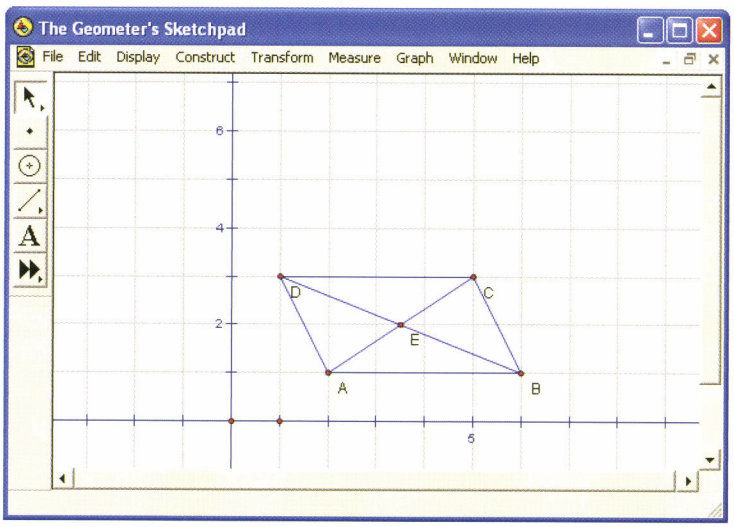
Technology Tip

The keyboard shortcut for the **Hide** option is Ctrl+H.

4. Construct line segments from C to D and from D to A.

5. Explain how you know that quadrilateral ABCD is a parallelogram.

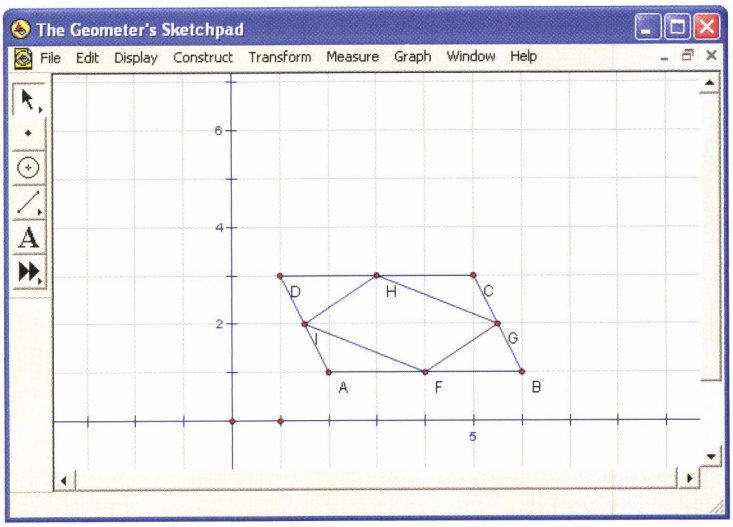
6. Construct diagonals AC and BD. Select the two diagonals and choose **Intersection** from the **Construct** menu. Measure the distance from the intersection point E to each of the four vertices. What can you conclude from these measurements?



7. Drag each of the vertices of ABCD to various new locations. Does ABCD remain a parallelogram? What do you notice about the lengths of AE, BE, CE, and DE?

8. **Reflect** What property do the diagonals of parallelograms have? Explain your reasoning.

9. Hide or delete the diagonals and their measurements. Construct the midpoint of each side of the parallelogram. Construct line segments to join the midpoints of adjacent sides. What properties does the quadrilateral formed by these line segments appear to have?



10. Make a conjecture about the type of quadrilateral formed by joining the midpoints of adjacent sides of a parallelogram.

11. Use measurements to determine whether your conjecture is true. Describe your findings.

12. Observe your measurements of quadrilateral FGHI as you drag the vertices of ABCD around the screen. Do any of the relationships among the measurements change?

13. **Reflect** What can you conclude about the midpoints of the sides of a parallelogram? Explain how you know that this conclusion applies to all parallelograms.

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

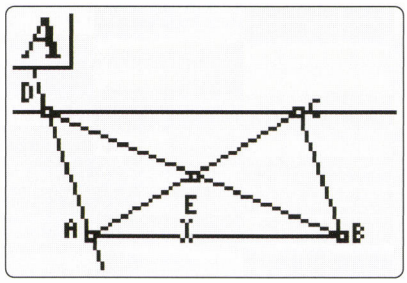
Method 3: Use a Graphing Calculator

1. Start the Cabri® Jr. application. Use the **F5** menu to show the axes if they do not appear on the screen. Choose **Segment** from the **F2** menu, and draw a line segment AB near the bottom of the screen. Draw another segment from B to a point C above the segment AB. Choose **Alpha-Num** from the **F5** menu. Label the points A, B, and C.

2. Choose **Parallel** from the **F3** menu. Move the cursor to segment AB until it flashes, and press ENTER. Select point C, segment BC, and point A in the same way.

3. Choose **Point** from the **F2** menu; then, choose **Intersection** from the submenu. Move the cursor to the line through point A and press ENTER. Then, move the cursor to the line through point C and press ENTER again. Label the intersection as point D.

4. Choose **Segment** from the **F2** menu, and construct diagonal line segments from A to C and from B to D. Then, choose **Intersection** from the **F2** menu, and construct the intersection of the line segments. Label this intersection as point E.



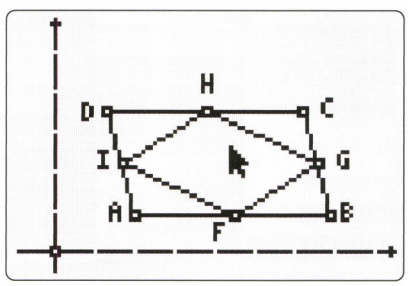
5. Choose **Measure** from the **F5** menu; then, choose **D. & Length**. Move the cursor to point A until it flashes and press ENTER. Select point E in the same way. Move the measurement to a convenient location. Press ENTER to lock the measurement in place. Use the same procedure to measure the lengths of BE, CE, and DE. What can you conclude from these measurements?

6. Press CLEAR. Move the cursor to point B and press ALPHA. Drag vertex B to various new locations. Does ABCD remain a parallelogram? What do you notice about the lengths of AE, BE, CE, and DE?

7. **Reflect** What property do the diagonals of parallelograms have? Explain your reasoning.

8. Press ALPHA. Choose **Hide/Show** from the **F5** menu; then, choose **Object**. Hide the diagonals and their measurements. Choose **Midpoint** from the **F3** menu. Select point A and then point B. Select point B again; then, select point C. Use the same procedure to construct the midpoints of CD and AD. Label the midpoints F, G, H, and I.

9. Construct line segments to join the midpoints of adjacent sides. What properties does the quadrilateral formed by these line segments appear to have?



10. Make a conjecture about the type of quadrilateral formed by joining the midpoints of adjacent sides of a parallelogram.

11. Use measurements to determine whether your conjecture is true. Describe your findings.

12. Observe your measurements of quadrilateral FGHI as you drag vertex B around the screen. Do any of the relationships among the measurements change?

13. **Reflect** What can you conclude about the midpoints of the sides of a parallelogram? Explain how you know that this conclusion applies to all parallelograms.

Example 1 Midpoints of a Quadrilateral

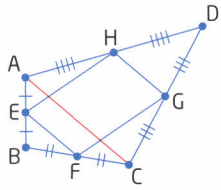
Draw any quadrilateral ABCD and find the midpoint of each side. Form a smaller quadrilateral EFGH inside the original one by drawing line segments joining the midpoints of adjacent sides. Investigate the properties of the smaller quadrilateral.

Solution

The inner quadrilateral EFGH appears to be a parallelogram. To test this conjecture, investigate the properties of the sides of EFGH.

Method 1: Use Pencil and Paper

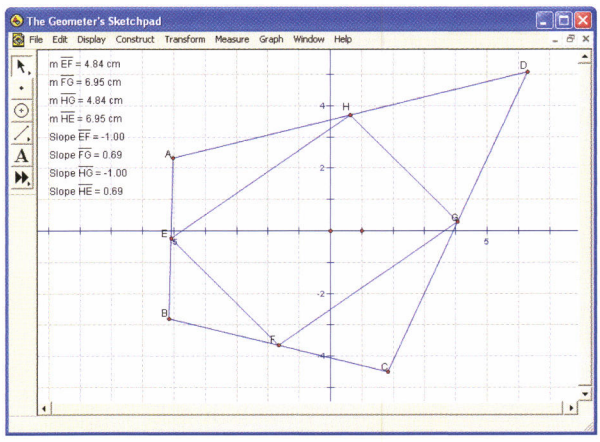
Draw the diagonal from vertex A to vertex C. In $\triangle DAC$, line segment GH joins the midpoints of sides AD and CD. Any line segment joining the midpoints of two sides of a triangle is parallel to the third side. Therefore, GH is parallel to CA. Similarly, EF joins the midpoints of two sides of $\triangle ABC$, so EF is also parallel to CA. Therefore, EF is parallel to GH.



Applying the same properties in $\triangle ABD$ and $\triangle BCD$ shows that EH is parallel to FG. Since both pairs of opposite sides in quadrilateral EFGH are parallel, it is a parallelogram.

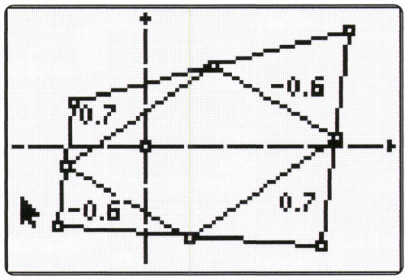
Method 2: Use Geometry Software

Construct any quadrilateral. Then, construct the midpoint of each side. Add line segments joining the midpoints of adjacent sides.



Measure the slope of each side of the inner quadrilateral. The measurements show that the opposite sides are parallel.

When the vertices of the original quadrilateral are dragged around the screen, the slopes of the opposite sides remain equal.



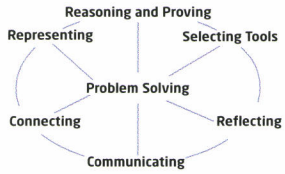
Therefore, the quadrilateral formed by joining the midpoints of adjacent sides of any quadrilateral is a parallelogram.

Did You Know?

The parallelogram formed by connecting the midpoints of adjacent sides of a quadrilateral is called a **Varignon parallelogram** after the French mathematician Pierre Varignon (1654–1722).

Example 2 Midpoints of a Trapezoid

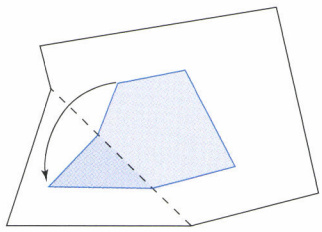
Investigate the properties of the midpoints of the non-parallel sides of a trapezoid.



Solution

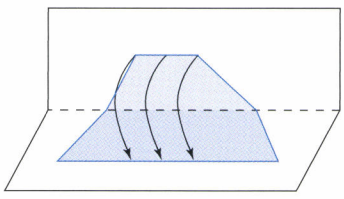
Method 1: Use Paper Folding

Draw a large trapezoid on a sheet of paper. Find the midpoint of each non-parallel side by folding your drawing so that one end is on top of the other.



Then, draw a line segment joining the midpoints of the non-parallel sides. This line segment appears to be parallel to the parallel sides of the trapezoid.

Now, fold the trapezoid along the line segment joining the midpoints of the non-parallel sides. The parallel sides of the trapezoid line up perfectly.



Therefore, the line segment joining the midpoints of the non-parallel sides is parallel to the other two sides and lies halfway between them.

Method 2: Use Analytic Geometry

Draw run-rise triangles on the two non-parallel sides. The sum of the runs of these sides is equal to the difference between the lengths of the parallel sides. So,

$QR = PS + run_{PQ} + run_{RS}$

Similarly, $TU = PS + run_{PT} + run_{US}$.

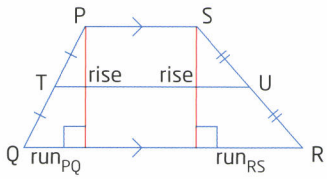
The run from the midpoint of a side to either endpoint is equal to half the run between the endpoints. So,

$$TU = PS + \frac{run_{QP}}{2} + \frac{run_{RS}}{2}$$

$$= \frac{2PS + run_{QP} + run_{RS}}{2}$$

$$= \frac{PS + (PS + run_{QP} + run_{RS})}{2}$$

$$= \frac{PS + QS}{2}$$



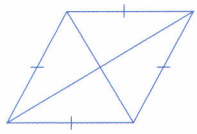
The length of the line segment joining the midpoints of the non-parallel sides is equal to the mean of the lengths of the parallel sides.

Key Concepts

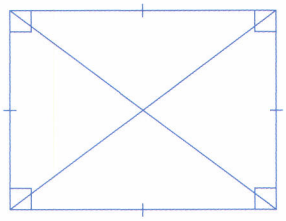
- The diagonals of a parallelogram bisect each other.
- Joining the midpoints of adjacent sides of any quadrilateral forms a parallelogram.
- The line segment joining the midpoints of the non-parallel sides of a trapezoid is parallel to the parallel sides and has a length equal to the mean of the lengths of the parallel sides.

Communicate Your Understanding

C1 Describe how you could fold a drawing to investigate the properties of the diagonals of a rhombus.



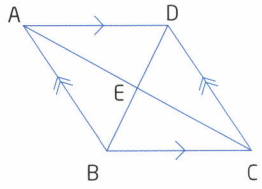
C2 Describe how you could use geometry software to investigate the properties of the diagonals of a rectangle.



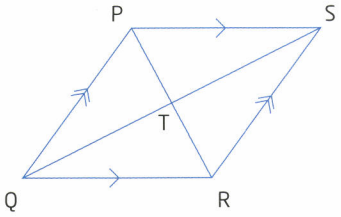
Practise

1. Which of the line segments inside each parallelogram are equal in length?

a)



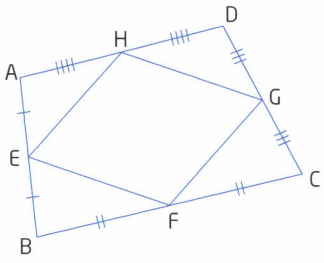
b)



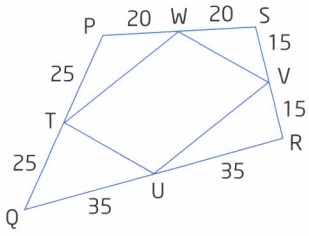
For help with questions 2 and 3, see Example 1.

2. Which of the line segments inside each quadrilateral are parallel?

a)



b)



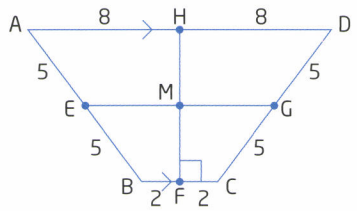
3. Which of the line segments inside each quadrilateral in question 2 are equal in length?

For help with questions 4 and 5, see Example 2.

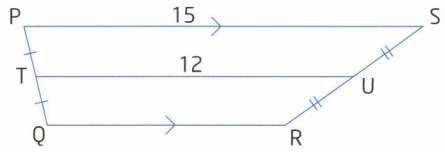
4. a) Which line segments are parallel in the figure below?

b) Find the length of EG.

c) Find the length of FH.



5. Find the length of QR.



Connect and Apply

6. Fold a drawing of a square to investigate the properties of its diagonals. Describe your findings.

7. Describe how you can use geometry software to determine when the diagonals of a parallelogram are equal in length.

8. a) Draw two line segments, AC and BD, that bisect each other at right angles. Then, draw a quadrilateral that has AC and BD as its diagonals. Classify the quadrilateral. Justify this classification.

b) Find the midpoints of AB, BC, CD, and DA, labelling these points E, F, G, and H, respectively. Draw line segments joining the midpoints of adjacent sides. Classify the quadrilateral EFGH. Justify this classification.

9. What properties make rectangles useful in the construction of buildings and other structures?

10. Sarah determines that the diagonals of a particular quadrilateral bisect each other and are equal in length. She concludes that the quadrilateral must be a square. Is Sarah correct? Explain your reasoning.

11. a) Investigate the properties of the diagonals of a rectangle. Describe your findings.

b) Investigate the properties of the diagonals of a kite. Describe your findings.

c) A rhombus is both a parallelogram and a kite. Make a conjecture about the properties of the diagonals of a rhombus.

d) Describe how you could use geometry software to test your conjecture in part c).

e) Make a table to summarize the properties of the diagonals of squares, rectangles, parallelograms, rhombi, and kites.

12. a) Predict the location of the balance point of a flat uniform rectangular object. Explain your reasoning.

b) Describe how you could determine if your prediction is correct.

13. Use a rectangle to estimate the location of the geographical centre of Canada.

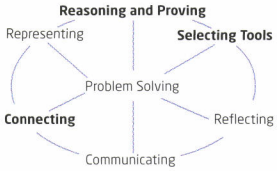
a) Trace the outline of Canada from the map on page 115 onto a sheet of paper. Draw a rectangle on your outline to approximate the shape of Canada as closely as you can.

b) Describe how to find the centre of this rectangle.

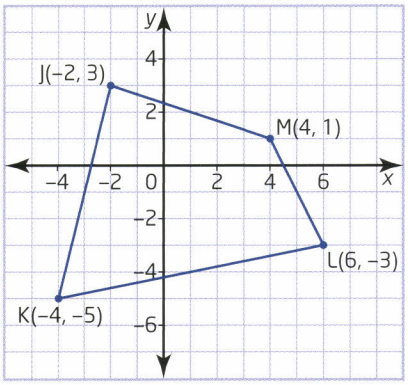
c) Find the centre you described in part b). Then, find the town or city closest to this centre. Compare your estimate of the geographical centre of Canada with those made by your classmates.

Achievement Check

14. Use analytic geometry to verify that



a) joining the midpoints of adjacent sides of quadrilateral JKLM forms a rhombus



b) the diagonals of the rhombus bisect each other at right angles

Extend

15. Investigate the Varignon parallelogram formed by joining the midpoints of adjacent sides of various types of quadrilaterals. Under what conditions is this parallelogram

a) a rectangle?

b) a square?

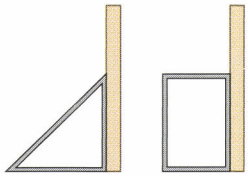
c) a rhombus?

Explain how you determined these conditions.

16. a) Can you draw a circle through all of the vertices of any rectangle? Justify your answer.

b) Can you draw a circle through all of the vertices of any quadrilateral? Support your answer with a diagram.

17. Explain why a triangular brace is stronger than a rectangular brace made of the same materials.



18. The English mathematician and theoretical physicist Roger Penrose (b. 1931) has studied the patterns that can be made with tiles that have shapes based on the golden ratio.

a) Use a library or the Internet to research Penrose tiles.

b) Draw the two rhombus shapes that Penrose used for his tiles. Explain how each shape is made from golden triangles.

c) Penrose tiles can cover a plane without any gaps, but the pattern is *aperiodic*. Explain what this term means.

d) Describe another unusual property of Penrose tiles.

e) Penrose also made aperiodic tilings using a particular kite and dart. Draw these two shapes, and explain how they are related to the golden ratio.

f) Make your own tile pattern with Penrose rhombi. Use computer software, or cut the tile shapes out of paper or cardboard.

WWW Go to www.mcgrawhill.ca/links/principles10 and follow the links to learn more about Penrose tilings.

19. Math Contest The golden ratio is $\varphi = \frac{1 + \sqrt{5}}{2}$. Show that $\frac{1}{\varphi} = \varphi - 1$.

3.4 Verify Properties of Quadrilaterals

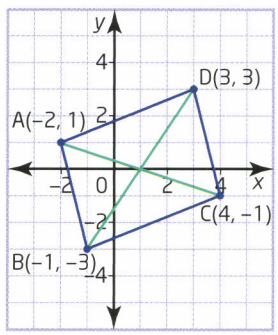
The new addition to the Royal Ontario Museum features a number of quadrilateral panels. This controversial design mimics the shape of mineral crystals.

Investigate

How can you verify properties of a parallelogram?

Tools
- grid paper

Method 1: Use Pencil and Paper



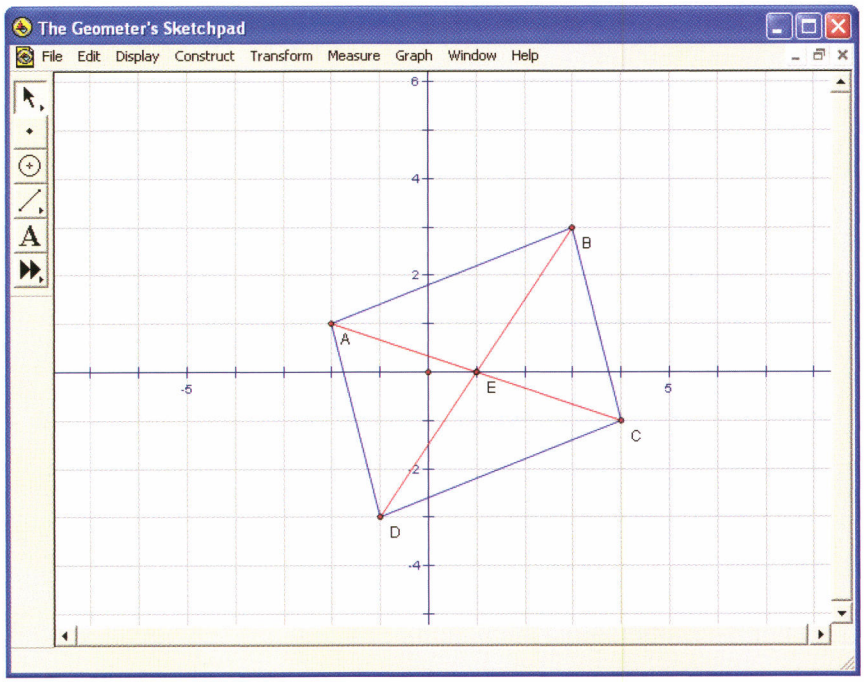
1. Calculate the slopes of the sides of quadrilateral ABCD. Explain how you can use these slopes to verify that ABCD is a parallelogram.
2. Calculate the length of each side of ABCD. Explain how you can use these lengths to show that ABCD is a parallelogram.
3. Explain how you can use angle measurements to show that ABCD is a parallelogram.
4. **Reflect** List the properties that you can use to determine whether a given quadrilateral is a parallelogram.
5. Verify that the diagonals, AC and BD, bisect each other.
6. Explain how you can use congruent triangles to verify that the diagonals of ABCD bisect each other.
7. **Reflect** Are all quadrilaterals with diagonals that bisect each other parallelograms? Can you use this property to determine whether a given quadrilateral is a parallelogram? Justify your answer.

Tools

- computer with *The Geometer's Sketchpad®*

Method 2: Use *The Geometer's Sketchpad®*

1. Construct the quadrilateral with vertices A(−2, 1), B(3, 3), C(4, −1), and D(−1, −3).



2. Measure the slope of each side of the quadrilateral. Explain how you can use these slopes to verify that ABCD is a parallelogram.

3. Measure the length of each side of ABCD. Explain how you can use these lengths to show that ABCD is a parallelogram.

4. Explain how you can use angle measurements to show that ABCD is a parallelogram.

5. **Reflect** List the properties that you can use to determine whether a given quadrilateral is a parallelogram.

6. Construct the diagonals AC and BD and their point of intersection. Verify that the diagonals bisect each other.

7. **Reflect** Are all quadrilaterals with diagonals that bisect each other parallelograms? Can you use this property to determine whether a given quadrilateral is a parallelogram? Justify your answer.

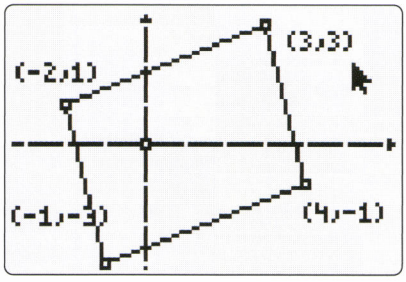
Method 3: Use a Graphing Calculator

Tools

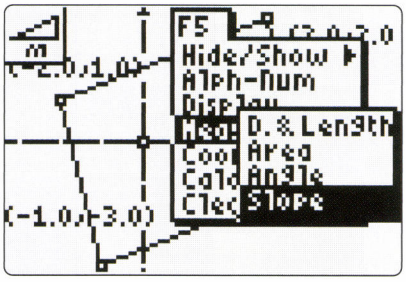
- TI-83 Plus or TI-84 Plus graphing calculator

1. Start the Cabri® Jr. application. Check that the axes are displayed.

2. Construct the quadrilateral with vertices A(−2, 1), B(3, 3), C(4, −1), and D(−1, −3). Choose **Coord. & Eq.** from the **F5** menu. Then, select the vertices to display their coordinates. Reposition any vertices that are not at the right coordinates.



3. Choose **Measure** from the **F5** menu, and **Slope** from the submenu. Measure the slope of each side of the quadrilateral. Explain how you can use these slopes to verify that ABCD is a parallelogram.



4. Measure the length of each side of ABCD. Explain how you can use these lengths to show that ABCD is a parallelogram.

5. Explain how you can use angle measurements to show that ABCD is a parallelogram.

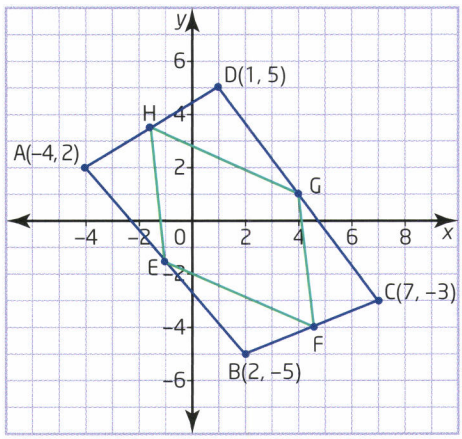
6. **Reflect** List the properties that you can use to determine whether a given quadrilateral is a parallelogram.

7. Construct the diagonals AC and BD and their point of intersection. Verify that the diagonals bisect each other.

8. **Reflect** Are all quadrilaterals with diagonals that bisect each other parallelograms? Can you use this property to determine whether a given quadrilateral is a parallelogram? Justify your answer.

Example 1 Midpoints of a Parallelogram

Verify that quadrilateral EFGH, formed by joining the midpoints of adjacent sides of quadrilateral ABCD, is a parallelogram.



Solution

The simplest way to verify that EFGH is a parallelogram is to show that the slopes of the opposite sides are equal. First, find the coordinates of the midpoint of each side of ABCD.

$$\text{E}(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{-4 + 2}{2}, \frac{2 + (-5)}{2}\right)$$
$$= (-1, -1.5)$$

$$\text{F}(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{2 + 7}{2}, \frac{-5 + (-3)}{2}\right)$$
$$= (4.5, -4)$$

$$\text{G}(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{7 + 1}{2}, \frac{-3 + 5}{2}\right)$$
$$= (4, 1)$$

$$\text{H}(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{1 + (-4)}{2}, \frac{5 + 2}{2}\right)$$
$$= (-1.5, 3.5)$$

Use these coordinates to find the slope of each side of EFGH.

$$m_{\text{EF}} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-4 - (-1.5)}{4.5 - (-1)}$$
$$= \frac{-2.5}{5.5} \times \frac{2}{2}$$
$$= -\frac{5}{11}$$

$$m_{\text{FG}} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{1 - (-4)}{4 - 4.5}$$
$$= \frac{5}{-0.5}$$
$$= -10$$

$$m_{\text{GH}} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3.5 - 1}{-1.5 - 4}$$
$$= \frac{2.5}{-5.5} \times \frac{2}{2}$$
$$= -\frac{5}{11}$$

$$m_{\text{HE}} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1.5 - 3.5}{-1 - (-1.5)}$$
$$= \frac{-5}{0.5}$$
$$= -10$$

Sides EF and GH have the same slope, so they are parallel. Similarly, FG is parallel to EH. Therefore, quadrilateral EFGH is a parallelogram.

Example 2 Properties of a Rhombus

a) Verify that the quadrilateral with vertices P(3, 3), Q(0, 1), R(3, −1), and S(6, 1) is a rhombus.

b) Verify that the diagonals of PQRS bisect each other at right angles.

Solution

a) Find the length of each side of PQRS.

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(0 - 3)^2 + (1 - 3)^2} \\ &= \sqrt{(-3)^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned} \qquad \begin{aligned} QR &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - 0)^2 + (-1 - 1)^2} \\ &= \sqrt{3^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} RS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - 3)^2 + (1 - (-1))^2} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned} \qquad \begin{aligned} PS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - 3)^2 + (1 - 3)^2} \\ &= \sqrt{3^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

All four sides are equal in length. Therefore, PQRS is a rhombus.

b) If the diagonals have the same midpoint, they bisect each other.

Find the coordinates of the midpoint of each diagonal.

For PR:

$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{3 + 3}{2}, \frac{3 + (-1)}{2}\right) \\ &= (3, 1) \end{aligned}$$

For QS:

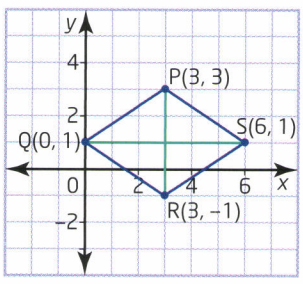
$$\begin{aligned} (x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{0 + 6}{2}, \frac{1 + 1}{2}\right) \\ &= (3, 1) \end{aligned}$$

I could show that the diagonals bisect each other by calculating and comparing the lengths PT, QT, RT, and ST.

Since the midpoints of the diagonals have the same coordinates, the diagonals bisect each other.

Now, calculate the slopes of the diagonals.

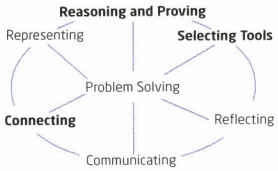
$$\begin{aligned} m_{PR} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - 3}{3 - 3} \\ &= \frac{-4}{0} \end{aligned} \qquad \begin{aligned} m_{QS} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - 1}{6 - 0} \\ &= 0 \end{aligned}$$



The graph of PQRS confirms that PR is vertical and QS is horizontal.

Since m_{PR} is undefined, PR is a vertical line segment. The zero value for m_{PR} indicates that QS is a horizontal line segment. So, PR and QS are perpendicular.

Therefore, PR and QS bisect each other at right angles.

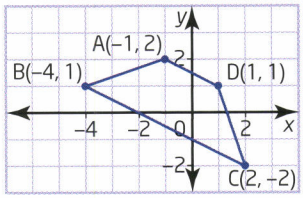


Key Concepts

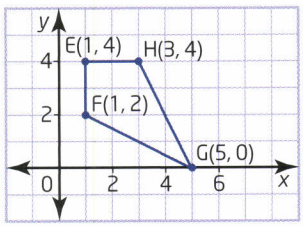
- You can use the formulas for lengths, midpoints, and slopes to verify properties of quadrilaterals.
- Often, there is more than one way to verify a property of a geometric shape.

Communicate Your Understanding

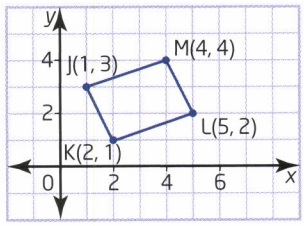
C1 Describe how to use analytic geometry to verify that quadrilateral ABCD is a trapezoid.



C2 Describe how to verify that the point of intersection of the diagonals of kite EFGH bisects only one of the diagonals.



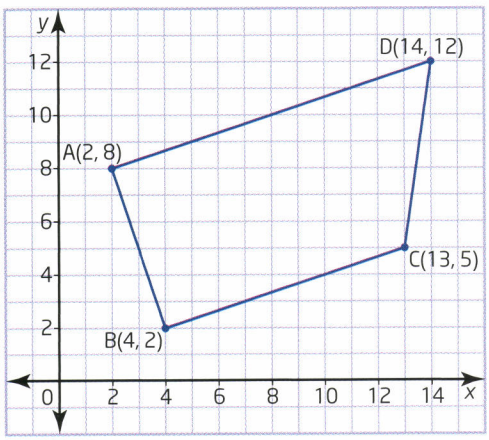
C3 Describe two methods for verifying that quadrilateral JKLM is a parallelogram. Which method is easier to use?



Practise

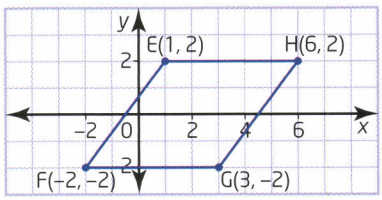
For help with question 1, see Example 1.

1. Verify that quadrilateral ABCD is a trapezoid.

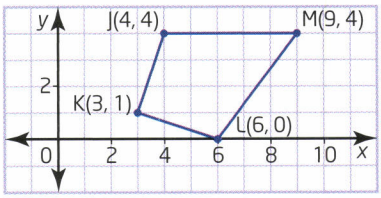


For help with questions 2 and 3, see Example 2.

2. Verify that quadrilateral EFGH is a rhombus.



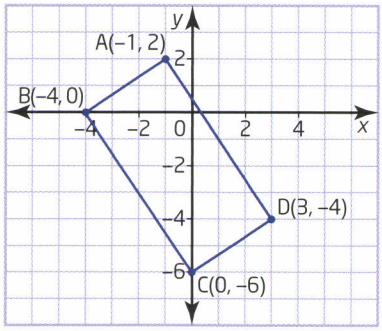
3. Verify that quadrilateral JKLM is a kite.



Connect and Apply

4. a) Verify that quadrilateral ABCD is a rectangle.

b) Verify that the diagonals of ABCD are equal in length and bisect each other.



5. a) Draw the quadrilateral with vertices P(0, 7), Q(−2, 1), R(4, −1), and S(6, 3).

b) Find the midpoint of each side. Join the midpoints of adjacent sides to form a new quadrilateral TUVW.

c) Verify that opposite sides of TUVW are parallel.

d) Verify that opposite sides of TUVW are equal in length.

6. Use Technology Use geometry software to answer question 5. Outline your method.

7. a) Draw the trapezoid with vertices A(−2, −2), B(2, −2), C(4, 1), and D(2, 4).

b) Verify that the line segment joining the midpoints of the non-parallel sides of the trapezoid is parallel to the other two sides.

8. Use Technology Use geometry software to answer question 7. Outline your method.

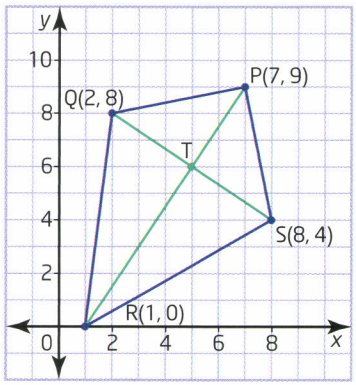
9. a) Verify that the diagonals of the rectangle with vertices J(−2, 1), K(2, 3), L(4, −1), and M(0, −3) bisect each other at right angles.

b) Do all rectangles have this property?

c) What can you conclude about the lengths of the sides of JKLM? Explain your reasoning.

10. a) Verify that PR bisects QS at right angles.

b) Verify that QS does not bisect PR.



11. a) Draw the quadrilateral with vertices A(3, 4), B(−1, 2), C(−3, −4), and D(5, −6). Then, join the midpoints of the adjacent sides of ABCD to form a new quadrilateral, EFGH.

b) Verify that EFGH is a rhombus.

c) Describe another method for verifying that EFGH is a rhombus.

12. a) Draw the quadrilateral with vertices P(−3, −1), Q(3, 1), R(7, 5), and S(1, 3). Then, draw the diagonals of PQRS.

b) Verify that the diagonals bisect each other.

c) What kind of quadrilateral is PQRS? Justify your answer.

13. Use Technology Use geometry software to answer question 12. Outline your method.

14. a) Draw the rhombus with vertices A(−5, 2), B(−1, 3), C(−2, −1), and D(−6, −2).

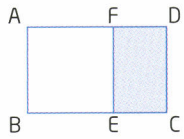
b) Verify that joining the midpoints of the adjacent sides of ABCD produces a rectangle.

15. Use Technology Use geometry software to answer question 14. Outline your method.

16. Chapter Problem

a) A rectangle with $\frac{\text{length}}{\text{width}} = \varphi$ is called a golden rectangle. On grid paper, draw a large rectangle, making the ratio of the length to the width as close to φ:1 as you can. The golden ratio, φ, equals 1.618....

b) Divide your golden rectangle into a square and a smaller rectangle, with the sides of the square equal to the width of the original rectangle. Measure the width and length of the smaller rectangle. Calculate the ratio of these dimensions.



c) Divide the smaller rectangle into a square and a third rectangle. Predict the ratio of the length and the width of the third rectangle. Measure these dimensions, and calculate their ratio. Do the measurements confirm your prediction?

d) Divide the third rectangle in the same way as the others to produce a fourth rectangle. Find the length-to-width ratio of this rectangle.

e) **Use Technology** Use *The Geometer's Sketchpad®* to construct a golden rectangle. Divide this rectangle into a square and a smaller rectangle. Divide the smaller rectangle in the same way. Continue the process, producing progressively smaller rectangles. Compare the length-to-width ratios of these rectangles. What can you conclude about these ratios?

f) Is the series of nested rectangles a fractal? Justify your answer.

g) Describe how you could use the nested rectangles to generate a golden spiral.

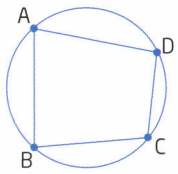
Extend

17. Let P(a, b), Q(c, d), R(e, f), and S(g, h) represent the coordinates of the vertices of a quadrilateral.

a) Determine the coordinates of the midpoints T, U, V, and W of sides PQ, QR, RS, and SP, respectively.

b) Verify that TUVW is a parallelogram.

18. Math Contest How are $\angle$BAD and $\angle$BCD related?



19. Math Contest How many four-digit numbers with no repeated digits can you make using the digits from 1 to 8?

A 7

B 8

C 1680

D 3584

E 4096

3.5 Properties of Circles

From the Canadian Arctic to Southern Africa, people have developed ingenious circular structures to use as homes or temporary shelters. The materials available and the climate greatly influence the design of these structures.

Investigate

What properties do circles have?

Tools

- circular object such as a juice can
- compasses
- ruler

Method 1: Use Pencil and Paper

1. Draw a circle on a sheet of paper by tracing around a circular object such as a juice can.
2. Find the centre of the circle by folding your drawing. Explain how you know where the centre is.
3. **Reflect** What property do diameters of a circle have?
4. Mark any two points on the circle and label them A and B. Join the points to form a **chord** of the circle. Draw the right bisector of this chord. What property does this right bisector have? Check whether the right bisectors drawn by your classmates have the same property.

chord

- line segment joining two points on a curve

5. Choose any point on a blank sheet of paper, but do not mark this point. Instead, mark three points that are all about the same distance from the first point. Label these three points P, Q, and R. Exchange your set of points with a classmate.
6. Find the centre of the circle that passes through the three points that your classmate marked. Explain your method. Draw the circle that passes through the points.
7. **Reflect** What property do the chords of a circle have?

Tools

- computer with *The Geometer's Sketchpad®*

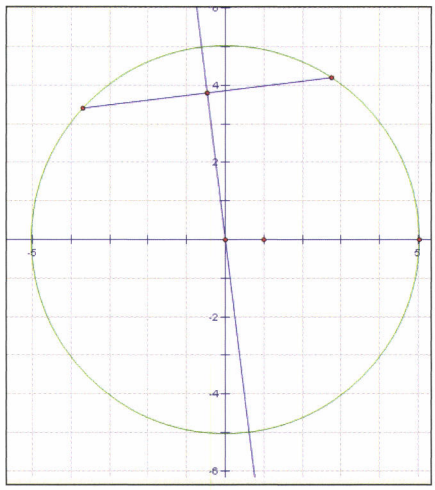
Method 2: Use *The Geometer's Sketchpad®*

1. Turn on the grid display. Construct a point at the centre of the grid and a second point near the edge. Using the first point as the centre, construct a circle by choosing **Circle by Centre and Point** from the **Construct** menu.

2. Construct any two points on the circle. Construct a **chord** by joining these two points with a line segment.

3. Construct the right bisector of the chord.

4. Select one of the endpoints of the chord. From the **Display** menu, choose **Animate Point**. As the endpoint of the chord moves around the circle, watch the right bisector of the chord. What property does this bisector have?



5. **Reflect** What property do the chords of a circle have? How you could use this property to find the centre of a circle given three points on the circumference?

Tools

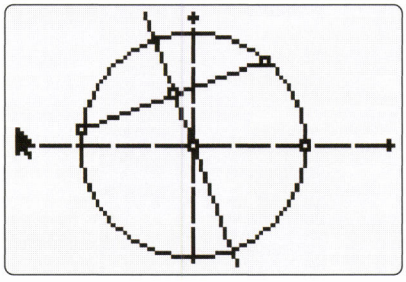
- TI-83 Plus or TI-84 Plus graphing calculator

Method 3: Use a Graphing Calculator

1. Start the Cabri® Jr. application. Check that the axes are displayed. Choose **Circle** from the **F2** menu. Place the centre near the middle of the screen, and place the end of the radius close to the top of the screen.

2. Choose **Segment** from the **F2** menu. Construct a **chord** by using a line segment to join any two points on the circle.

3. Choose **Perp. Bis.** from the **F3** menu, and construct the right bisector of the chord.

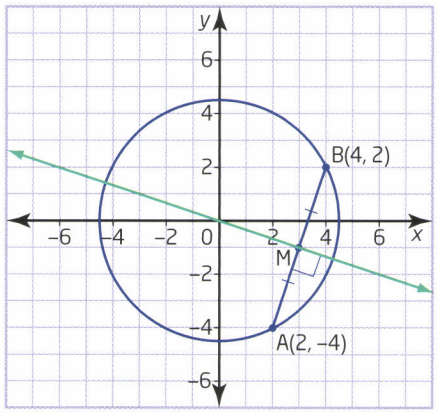


4. Move the cursor to one endpoint of the chord, and press ALPHA. Watch the right bisector of the chord as you move the endpoint around the circle. What property does this bisector have?

5. **Reflect** What property do the chords of a circle have? How could you use this property to find the centre of a circle given three points on the circumference?

Example 1 Right Bisector of a Chord

Verify that the centre of this circle lies on the right bisector of the chord AB.



Solution

The centre lies on the right bisector of AB only if the coordinates (0, 0) satisfy the equation of the right bisector.

Understand the Problem

The midpoint of AB is on the right bisector. Use the coordinates of A and B to find the midpoint and the slope of AB. Then, calculate the slope of a line perpendicular to AB. Use this slope and the coordinates of the midpoint, M, to determine the equation of the right bisector of AB.

Choose a Strategy

Find the midpoint coordinates and the slope of AB.

Carry Out the Strategy

$$\mathrm{M}(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{2 + 4}{2}, \frac{-4 + 2}{2}\right)$$
$$= (3, -1)$$

$$m_{\mathrm{AB}} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - (-4)}{4 - 2}$$
$$= \frac{6}{2}$$
$$= 3$$

Since the right bisector is perpendicular to AB, the slope of this bisector is the negative reciprocal of m_{AB}.

$$-\frac{1}{m_{\mathrm{AB}}} = -\frac{1}{3}$$

Use this slope and the coordinates M(3, −1) to find the y-intercept.

$$y = mx + b$$
$$-1 = -\frac{1}{3}(3) + b$$
$$-1 = -1 + b$$
$$0 = b$$

All lines with a y-intercept of 0 pass through the origin.

The equation of the right bisector is $y = -\frac{1}{3}x$. The coordinates (0, 0) satisfy this equation. Therefore, the centre of the circle lies on the right bisector of the chord PQ.

Reflect

Example 2 Points on a Circle

a) Show that the points P(9, −3), Q(8, 6), and R(−1, 5) lie on a circle with its centre at C(4, 1).

b) Does any other circle pass through points P, Q, and R? Explain.

Solution

a) If the three points lie on a circle centred at (4, 1), each point must be the same distance from (4, 1). Compare the lengths of CP, CQ, and CR.

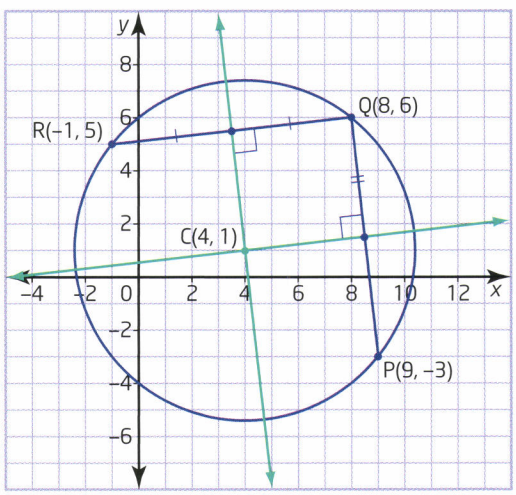
$$\begin{aligned} CP &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(9 - 4)^2 + (-3 - 1)^2} \\ &= \sqrt{5^2 + (-4)^2} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} CQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(8 - 4)^2 + (6 - 1)^2} \\ &= \sqrt{4^2 + 5^2} \\ &= \sqrt{41} \end{aligned}$$

$$\begin{aligned} CR &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-1 - 4)^2 + (5 - 1)^2} \\ &= \sqrt{(-5)^2 + 4^2} \\ &= \sqrt{41} \end{aligned}$$

Since CP = CQ = CR, the points P, Q, and R all lie on a circle centred at C(4, 1).

b) The right bisector of PQ includes all points that are equidistant from P and Q. Similarly, the right bisector of QR includes all points that are equidistant from Q and R. These two lines meet only at point C(4, 1). There is no other point equidistant from P, Q, and R.



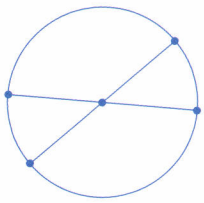
Therefore, the circle with centre C(4, 1) and radius $\sqrt{41}$ is the only circle that passes through the points P, Q, and R.

Reflect

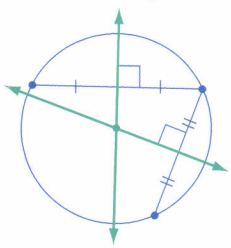
You cannot determine the centre of a circle from just two points on the circumference. The centre can lie anywhere on the right bisector of the chord between the two points.

Key Concepts

- The diameters of a circle intersect at the centre of the circle.



- The right bisector of a chord of a circle passes through the centre of the circle.
- The right bisectors of two chords of a circle intersect at the centre of the circle.
- There is only one circle that passes through three given non-collinear points.



Communicate Your Understanding

C1 Describe how to find the point that is equidistant from three given points by

a) folding a plot of the given points

b) constructing lines

c) using analytic geometry

C2 Describe how you would determine the balance point for a flat circular object.

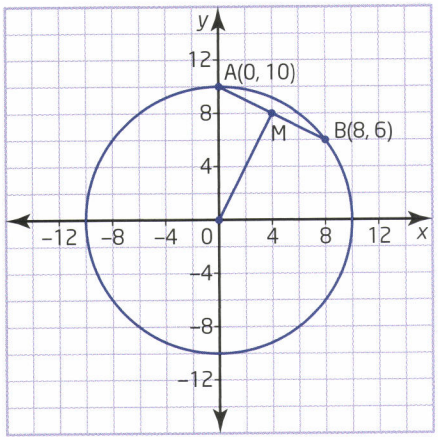
Practise

For help with question 1, see Example 1.

1. a) Find the coordinates of the midpoint, M, of AB.

b) Find the slope of the chord AB.

c) Verify that OM is perpendicular to AB.



For help with questions 2 to 4, see Example 2.

2. a) Verify that the points P(−1, −2), Q(2, 7), and R(6, 5) are equidistant from the point C(2, 2).

b) Draw the circle that passes through points P, Q, and R.

3. a) Verify that the points A(12, 6), B(4, 10), and C(0, 2) lie on a circle with its centre at D(6, 4).

b) Determine the length of the radius of the circle.

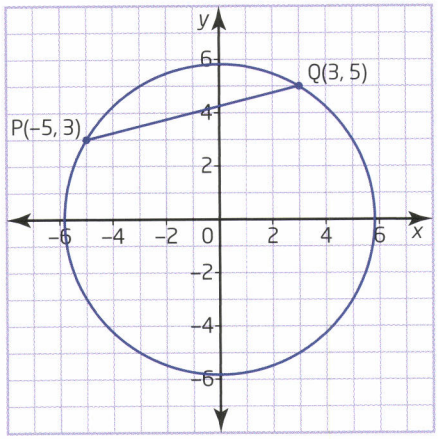
c) Plot points A, B, and C on grid paper, and draw the circle that passes through the points. Use your drawing to check your answers to parts a) and b).

4. a) Verify that the points E(−5, 0), F(−2, 3), and G(6, −11) lie on a circle with its centre at H(2, −4).

b) Determine the length of the radius of the circle.

Connect and Apply

5. Verify that the centre of this circle lies on the right bisector of the chord PQ.



6. a) Explain how you know that the origin is the centre of the circle defined by the equation $x^2 + y^2 = 45$.

b) Verify that the points R(−3, 6) and S(−6, −3) lie on the circle.

c) Verify that the line through the origin and the midpoint of the chord RS is perpendicular to the chord.

7. A machinist needs to drill a hole in the centre of a circular part. Describe how the machinist could mark the correct location for this hole.

8. a) You have 3.0 m of edging to put around a flower bed. Find the maximum area you can enclose if the shape of the flower bed is an equilateral triangle.

b) Find the maximum area you can enclose if the flower bed is square.

c) Find the maximum area you can enclose if the flower bed is circular.

d) What property of circles makes them a useful shape for the base of storage tanks and some types of buildings?

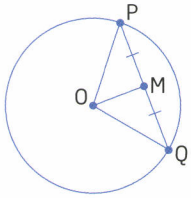
9. Find the centre of the circle that passes through the points A(−7, 4), B(−4, 5), and C(0, 3).

10. **Use Technology** Use geometry software to answer question 9. Outline your method.

11. Three friends live in Sudbury, Toronto, and Windsor. They are planning to go camping together and want to find a park that is approximately the same distance from each of their homes. Describe how the friends could fold an Ontario roadmap to help them find a suitable campground.

12. On a town map, the coordinates of three schools are J(8, 13), K(10, 7), and L(14, 15). The town is planning to build a new swimming pool that is the same distance from all three schools. Determine the coordinates for the pool.

13. Draw a circle with centre O. Add any chord PQ, with midpoint M. What can you conclude about $\triangle OMP$ and $\triangle OMQ$? Explain your reasoning.



14. Draw any circle. Draw a diameter of the circle and label its endpoints J and K. Let L be any other point on the circumference of the circle. Use angle sums in triangles to show that $\triangle JKL$ is a right triangle.

15. **Use Technology** Use geometry software to answer question 14. Outline your method.

Extend

16. To find a good location for a community hospital, planners could find the smallest circle that encloses all the homes on a map of the community.

a) What is the advantage of a location at the centre of the smallest enclosing circle?

b) What other factors might prevent the centre from being the best location for the hospital?

c) Using grid paper or geometry software, plot 15 points to represent neighbourhoods. Try to find the smallest circle that encloses these points. Describe the method that you used. Mark where you would place the hospital for the neighbourhoods that the dots represent. Explain why you chose this location.

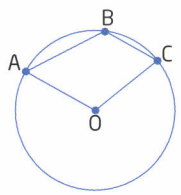
17. A pilot filed a flight plan that listed a cruising speed of 160 km/h with enough fuel on board for 3.5 h of flying. After 2 h, the aircraft passed over Lake Traverse. The aircraft was reported missing when it failed to reach its planned destination.

a) Sketch a diagram showing the area where the plane may have crashed.

b) How large is this area?

c) Which part of the search area should be searched first? Explain your reasoning.

18. **Math Contest** Show how $\angle ABC$ and $\angle AOC$ in quadrilateral ABCD are related.



19. **Math Contest** Keshawn and Samantha are in a science class with 18 other students. The teacher randomly divides the class into 10 pairs of laboratory partners. The probability that Keshawn and Samantha are laboratory partners is

A $\frac{1}{10}$

B $\frac{1}{19}$

C $\frac{1}{20}$

D $\frac{1}{190}$

E $\frac{1}{380}$

Chapter 3 Review

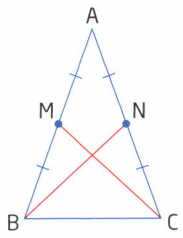
3.1 Investigate Properties of Triangles, pages 110–116

1. a) Define a median.

b) List two additional properties of the medians of a triangle.

c) Outline how you could use geometry software to show that the medians of all triangles have these properties.

2. a) Use congruent triangles to show how the lengths of the medians to the two equal sides of an isosceles triangle are related.



b) Describe how you can use geometry software to demonstrate that this relationship applies for all isosceles triangles.

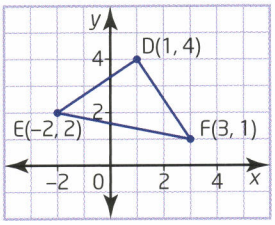
3. a) Draw a right triangle with the shorter sides aligned with the coordinate axes. Then, draw the right bisectors of all three sides.

b) Show that the point of intersection of the right bisectors of the sides of any right triangle lies on the hypotenuse.

c) Describe how you can use geometry software to answer part b).

3.2 Verify Properties of Triangles, pages 117–127

4. a) Verify that $\triangle DEF$ is a right triangle.



b) Describe another method that you could use to answer part a).

5. a) Verify that the altitude from vertex J bisects side KL in the triangle with vertices J(−5, 4), K(1, 8), and L(−1, −2).

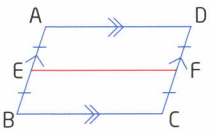
b) Classify $\triangle JKL$. Explain your reasoning.

3.3 Investigate Properties of Quadrilaterals, pages 128–136

6. List two properties of the diagonals of each geometric shape.

a) square **b)** parallelogram **c)** kite

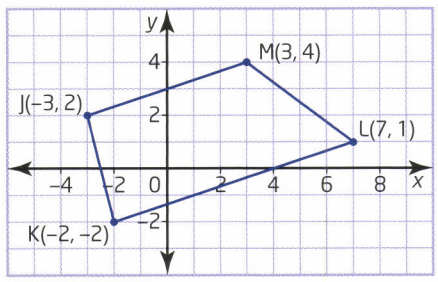
7. a) Show that the line segment joining the midpoints of opposite sides of any parallelogram bisects the area of the parallelogram.



b) Describe another way to answer part a).

3.4 Verify Properties of Quadrilaterals, pages 137–144

8. Verify that quadrilateral JKLM is a trapezoid.



9. a) Classify the quadrilateral with vertices T(2, 4), U(8, 2), V(7, −1), and W(1, 1). Justify this classification.

b) Verify a property of the diagonals of TUVW.

3.5 Properties of Circles, pages 145–151

10. a) Show that A(−12, −5) and B(12, 5) are endpoints of a diameter of the circle defined by $x^2 + y^2 = 169$.

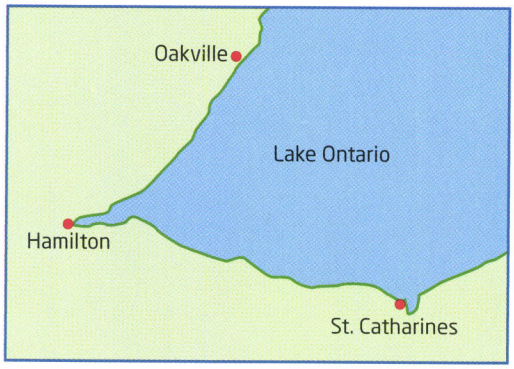
b) State the coordinates of another point, C, on the circle.

c) Show that △ABC is a right triangle.

11. a) Verify that points P(5, 7) and Q(7, −5) lie on the circle with equation $x^2 + y^2 = 74$.

b) Verify that the right bisector of the chord PQ passes through the centre of the circle.

12. Divers from scuba clubs in St. Catharines, Hamilton, and Oakville are working together to practise rescues in deep water. Describe how the clubs could find a practice site that is equidistant from the three towns.



Chapter Problem Wrap-Up

The golden ratio is also known as the divine proportion, the golden mean, and the golden section. Many people find that objects with proportions in the golden ratio are pleasing to the eye. For this reason, artists and architects sometimes use these proportions in their work.

a) Research the golden ratio at a library or on the Internet. Is the golden ratio a rational number? Explain.

b) Give an example of a regular geometric shape that involves the golden ratio. Describe how the golden ratio appears in this shape.

c) Give an example of a natural object that may contain the golden ratio. Explain why researchers disagree on whether the golden ratio occurs frequently in nature.

d) Some researchers argue that designers in ancient times used the golden ratio in buildings such as the Parthenon in Athens, Greece, and the Great Pyramid at Giza, Egypt. Do you think these claims are valid? Justify your answer.

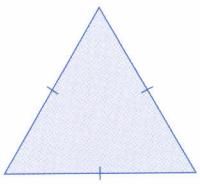
e) Measure a textbook. Are any of the dimensions related by the golden ratio? Explain.

WWW Go to www.mcgrawhill.ca/links/principles10 and follow the links to learn more about the golden ratio.

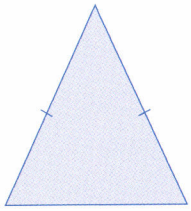
Chapter 3 Practice Test

1. Which of these triangles have at least two medians that are equal in length? Justify your choices.

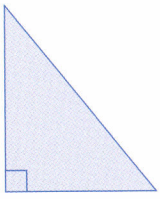
A



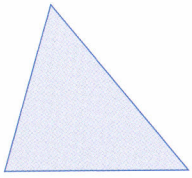
B



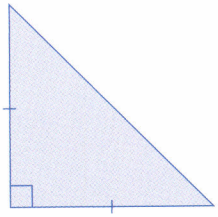
C



D



E



2. Which of the triangles in question 1 have a median that is also an altitude? Justify your choices.

3. Sketch an example of each of these types of quadrilateral. Show the diagonals on each sketch and indicate whether they are equal in length and whether they bisect each other.

a) square

b) rectangle

c) parallelogram

d) rhombus

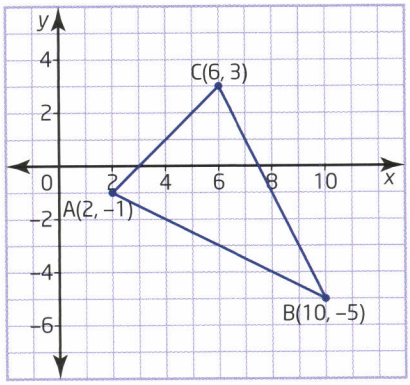
e) trapezoid

f) quadrilateral with no equal sides

4. Show that two of the altitudes of an isosceles triangle are equal in length.

5. a) Verify that $\triangle ABC$ is isosceles.

b) Verify that the centroid of $\triangle ABC$ lies at $(6, -1)$.

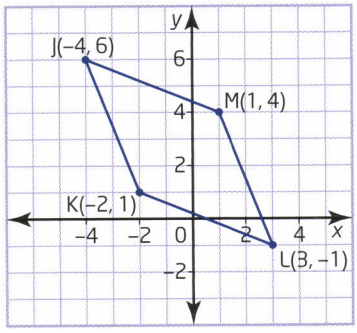


6. a) Show that the triangle with vertices $D(-2, 5)$, $E(-4, 1)$, and $F(2, 3)$ is a right triangle.

b) Verify that the midpoint of the hypotenuse of $\triangle DEF$ is equidistant from all three vertices.

7. a) Use analytic geometry to verify that quadrilateral JKLM is a rhombus.

b) Describe how to use geometry software to answer part a).



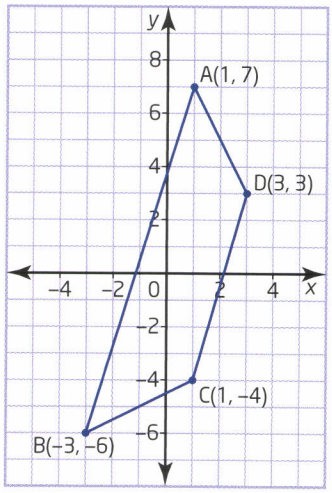
8. a) Find the midpoint of each side of the quadrilateral with vertices $P(-3, 8)$, $Q(1, 10)$, $R(5, 6)$, and $S(7, -4)$.

b) Show that joining the midpoints of the adjacent sides of PQRS forms a parallelogram.

9. a) Verify that C(5, 2) is the centre of the circle that passes through points T(5, 15), U(17, −3), and V(−8, 2).

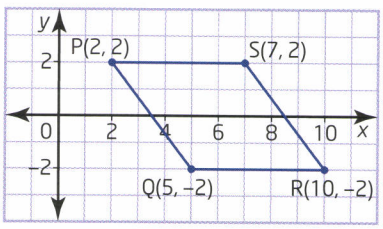
b) Find the radius of the circle.

10. Verify that quadrilateral ABCD is not an isosceles trapezoid.



11. Describe how to use geometry software to answer question 10.

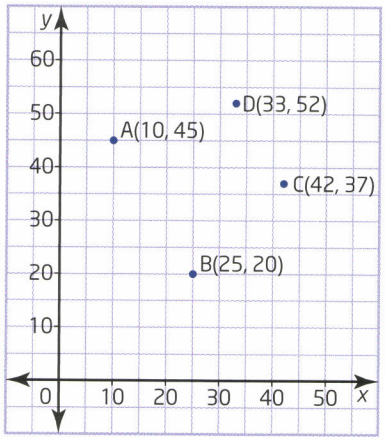
12. a) Verify that quadrilateral PQRS is a rhombus.



b) Verify that the diagonals of PQRS bisect each other.

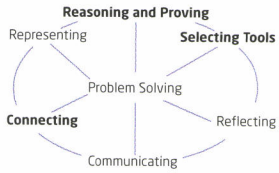
c) Verify that the diagonals of PQRS meet at right angles.

13. A new hospital will serve the four small towns shown on the map. Where would you build the hospital? Justify this location.



Achievement Check

14. The vertices of △EFG are E(−3, 5), F(0, −1), and G(6, 5).



a) Find the coordinates of the point of intersection of the medians of △EFG.

b) Find the coordinates of the point of intersection of the right bisectors of the sides of △EFG.

c) Find the coordinates of the point of intersection of the altitudes of △EFG.

d) Verify that these three points of intersection are collinear.

Chapters 1 to 3 Review

Chapter 1 Linear Systems

1. Solve each linear system. For each system, explain why you chose the method that you used. Check each solution.

a) $x + 2y = 3$
$3x - y = 1$

b) $x - y = -1$
$2x + y = 4$

c) $3x + 2y = 28$
$5x - 3y = 15$

d) $\frac{1}{2}x + y = 4$
$x + \frac{1}{3}y = 2$

2. All 120 seats in a hall were filled for a concert. The tickets cost \$10 for adults and \$6 for students. The total proceeds were \$980. How many adults attended the concert? How many students?

3. Two wind turbines generate a total of 57 kW of power. The larger turbine generates twice as much power as the smaller one. Find the power output of each turbine.

4. A contractor rented a sander for 5 h and a polisher for 5 h for \$50. For another job, she rented the sander for 4 h and the polisher for 8 h. The rental fees for that job totalled \$56. Find the hourly rate charged for each tool.

5. Lou's class is selling T-shirts for a fundraiser. The supplier charges \$750 for the initial design and set-up plus \$5 for each imprinted T-shirt. The students sell the T-shirts for \$15 each.

a) How many T-shirts do the students need to sell to break even?

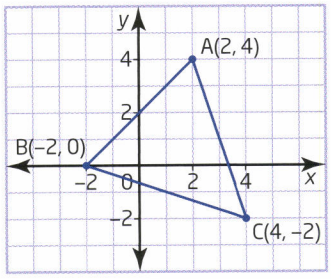
b) How much profit will the students make if they sell 150 T-shirts?

6. For his coffee shop, Abdul wants to make a mocha-java blend that will sell for \$18/kg. The mocha coffee beans sell for \$20/kg, and the java coffee beans sell for \$15/kg. How many kilograms of each kind of coffee bean should he use to make 50 kg of the mocha-java blend?

Chapter 2 Analytic Geometry

7. a) Determine the midpoint of each side of $\triangle ABC$.

b) Calculate the perimeter of $\triangle ABC$.



8. a) Plot the triangle with vertices D(6, 8), E(1, 8), and F(4, 2).

b) Determine the equation of the right bisector of side DE.

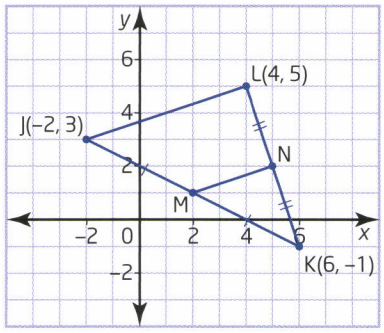
c) Determine the equation of the right bisector of side EF.

d) Determine the coordinates of the point of intersection, M, of the right bisectors in parts a) and b).

e) Show that point M is equidistant from vertices D, E, and F.

9. Classify the triangle with vertices G(−4, −1), H(2, −3), and I(4, 3). Explain your reasoning.

10. M and N are the midpoints of sides JK and KL. Show that MN is parallel to JL.



11. Find an equation for the circle that is centred at the origin and

a) has a radius of 7

b) has a radius of $\sqrt{10}$

c) passes through the point $(-5, 12)$

12. On the plan for a new play area in a park, a paddling pool is represented by a circle with the equation $x^2 + y^2 = 16$. Dimensions on the plan are in metres. What will the perimeter of the pool be?

Chapter 3 Geometric Properties

13. a) Draw the triangle with vertices P(−2, −2), Q(2, 4), and R(8, 0).

b) Show algebraically that △PQR is a right triangle.

c) Is △PQR also an isosceles triangle? Use algebraic reasoning to justify your answer.

d) Determine the area of △PQR.

14. Use Technology Use *The Geometer's Sketchpad®* or Cabri® Jr. to verify your answers to question 12. Outline your method.

15. a) Plot the triangle with vertices A(−1, 1), B(3, 5), and C(5, −1).

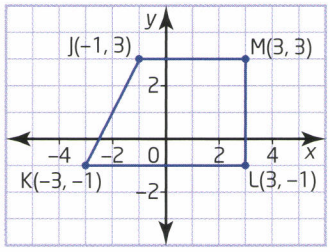
b) What type of triangle does △ABC appear to be?

c) Use analytic geometry to verify your answer in part a).

d) Verify that the median from C to AB is also an altitude of the triangle.

16. Use analytic geometry to verify that D(−1, 3), E(2, 2), F(3, −1), and G(0, 0) are the vertices of a parallelogram.

17. JKLM is a trapezoid. P, Q, R, and S are the midpoints of the sides JK, KL, LM, and MJ, respectively. What type of quadrilateral is PQRS? Use analytic geometry to verify your answer.



18. The vertices of rhombus TUVW are T(−3, 2), U(2, 2), V(0, −2), and W(5, −2).

a) Use analytic geometry to show that the diagonals of TUVW are right bisectors of each other.

b) Use *The Geometer's Sketchpad®* or Cabri® Jr. to verify the property in part a). Outline your method.

Tasks

Multiple Midpoints

Can you determine the coordinates of the vertices of a triangle given just the coordinates of the midpoints of its sides?

a) Start with the midpoints P(0, 0), Q(0, 4), and R(4, 0). Find the coordinates of the vertices of a $\triangle ABC$ that has P, Q, and R as the midpoints of its sides.

b) Next, find the vertex coordinates of a $\triangle DEF$ that has P(0, 0), Q(2, 4), and R(4, 0) as the midpoints of its sides.

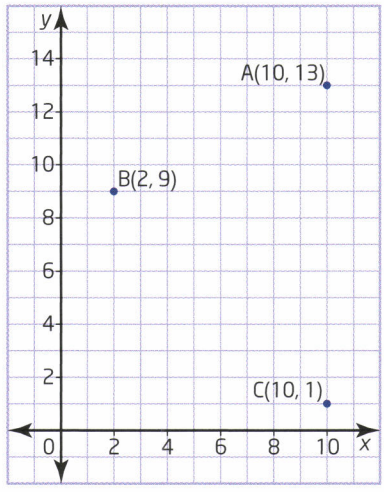
c) Choose three midpoints yourself. Find the vertex coordinates of a $\triangle GHI$ that has these three points as the midpoints of its sides.

d) Describe the steps for finding the vertex coordinates of a $\triangle JKL$ that has P(0, 0), Q(k, m), and R(p, q) as the midpoints of its sides.

e) Is it always possible to find the vertices for such a triangle? Explain.

A Site for the New Hospital

A new hospital is being planned to serve three towns: Abbott, Banting, and Colton. On a map, the coordinates of the towns are A(10, 13), B(2, 9), and C(10, 1), respectively.



a) One proposal is to build the hospital at the point that is equidistant from the three towns.

- Use analytic methods to determine the coordinates of this point.
- Verify that this point is the same distance from each town.
- List some reasons why this point might not be the best site for the new hospital.

b) Prepare a letter to the area's planning council describing how to choose a good location. Use mathematical reasoning to justify the method you recommend.

c) Another proposal is to choose the location that has the shortest mean driving distance to the hospital for the residents of the three towns. Colton has twice the population of either Abbott or Banting. Therefore, the location for the hospital should be equidistant from Abbott and Banting, but half as far away from Colton. Use *The Geometer's Sketchpad*® to find the coordinates of this location. (Note: Analytic geometry can be used to find a "weighted" centre. However, the algebra for this method is quite challenging.)

Pythagoras Park

The mathematics department of a university is sponsoring a competition to design a new city park. The park will be constructed on a rectangular lot beside the department's offices. The lot is 250 m by 400 m.

To encourage applications of mathematics to landscaping, the rules for the competition require that all designs include at least five of the following geometric shapes:

- Triangles: equilateral, isosceles, right, scalene
- Quadrilaterals: square, rectangle, parallelogram, rhombus, trapezoid, kite
- Circle, semicircle

The designs must also have at least five of the following features:

- median
- altitude
- angle bisector
- right bisector
- centroid
- orthocentre
- incentre
- circumcentre
- parallel lines
- perpendicular lines
- diagonal
- chord

Form a design team with two of your classmates to enter the competition. Using grid paper, prepare a fully labelled scale diagram of your design. Write a description of the features of your design. The scale diagram and written description should include the location of any gardens, trees, walkways, lights, benches, fountains, gazebos, picnic areas, playgrounds, or other features.